Multivariate Regression Concepts

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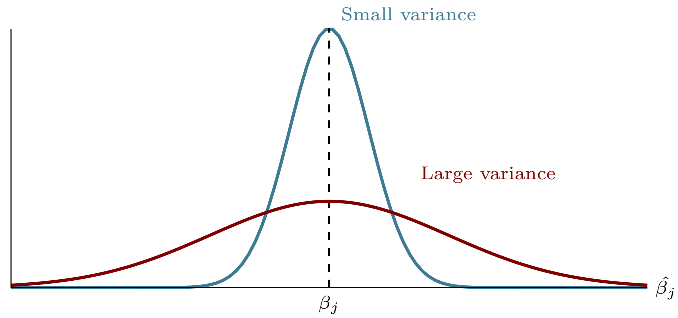
ECON 480 - Econometrics - Final Exam

Multivariate Regression

- Omitted Variable Bias
  - A variable Z causes omitted variable bias if:
    1. $corr(X, Z) \neq 0$, X and Z are correlated
    2. $corr(Z, Y) \neq 0$, Z is in the error term that explains Y
  - Omitted variable bias can be avoided by including Z in the regression (as X_2)
- Multivariate Regression Model
$$\widehat{Y_i} = \hat{\beta}_0 + \hat{\beta}_1 X_{1i} + \hat{\beta}_2 X_{2i} + \hat{\epsilon}_i$$
  - $\hat{\beta}_0$: predicted value of $\hat{Y}_i$ when $X_{1i} = 0; X_{2i} = 0$
  - $\hat{\beta}_1 = \frac{\Delta Y_i}{\Delta X_{1i}}$, marginal effect of X_{1i} on Y_i, holding X_{2i} constant
  - $\hat{\beta}_2 = \frac{\Delta Y_i}{\Delta X_{2i}}$, marginal effect of X_{2i} on Y_i, holding X_{1i} constant
- Measuring Omitted Variable Bias
  - Suppose we omit X_{2i} and run an Omitted Regression
$$Y_i = \alpha_0 + \alpha_1 X_{1i} + \nu_i$$
  - If we run an Auxiliary Regression of X_{2i} on X_{1i}:
$$X_{2i} = \delta_0 + \delta_1 X_{1i} + \tau_i$$
    * Size and significance of δ_1 measures relationship between X_{1i} and X_{2i}
$$\alpha_1 = \beta_1 + \beta_2 \delta_1$$
  - Biased estimate α_1 in Omitted Regression picks up:
    * True effect of X_{1i} on Y_i (β_1)
    * Effect of X_{2i} on Y_i (β_2) as pulled through the relationship between X_{1i} and X_{2i} (δ_1)
  - Conditions for Z being an omitted variable
    * Z_i must be a determinant of Y_i ($\beta_2 \neq 0$)
    * Z_i is correlated with X_{1i} ($\delta_1 \neq 0$)
- Variance of OLS estimators $\hat{\beta}_j$
$$var[\hat{\beta}_j] = \frac{1}{(1 - R_j^2)} * \frac{\hat{\sigma}^2}{n \times var[X_j]}$$
  and Standard error
$$s.e.[\hat{\beta}_j] = \sqrt{var[\hat{\beta}_j]}$$



$\hat{\beta}_j$ is a random variable, so it has its own sampling distribution with mean $E[\hat{\beta}_j]$ and standard error $se[\hat{\beta}_j]$

- Affected by 4 major factors:
  1. Model fit, where SER=$\hat{\sigma}$
  2. Sample size n
  3. Variation in X_j
  4. Variance Inflation Factor (VIF) $\frac{1}{1-R_j^2}$
     - Independent variables are **multicollinear** if they are correlated

       $$corr(X_j, X_l) \neq 0 \text{ for } j \neq l$$

     - Does not bias estimators, but increases their variance & standard errors
     - R_j^2 is the R^2 from an auxiliary regression of X_j on all other regressors
     - VIF quantifies how by many times the variance of $\hat{\beta}_j$ increased because of multicollinearity
       * $VIF > 10$ (or $\frac{1}{VIF} > 0.10$) is bad
     - **Perfect multicollinearity** when a regressor is an exact linear function of (an)other regressor(s) – cannot run a regression, a logical impossibility

       $$|corr(X_1, X_2)| = 1$$

Dummy Variables

- Dummy variable

$$D_i = \begin{cases} 1 & \text{if } i \text{ meets condition} \\ 0 & \text{if } i \text{ does not meet condition} \end{cases}$$

- Dummy variables measure group means

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 D_i$$

  - When $D_i = 0$ (Control group):
    - $\hat{Y}_i = \hat{\beta}_0$
    - $E[Y|D_i = 0] = \hat{\beta}_0 \iff$ the mean of Y when $D_i = 0$
  - When $D_i = 1$ (Treatment group):
    - $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 D_i$
    - $E[Y|D_i = 1] = \hat{\beta}_0 + \hat{\beta}_1 \iff$ the mean of Y when $D_i = 1$
  - Difference in group means:

$$\begin{aligned} &= E[Y_i|D_i = 1] - E[Y_i|D_i = 0] \\ &= (\hat{\beta}_0 + \hat{\beta}_1) - (\hat{\beta}_0) \\ &= \hat{\beta}_1 \end{aligned}$$

- Transforming categorical variables into dummies
  - A categorical variable (e.g. region, class standing, etc) can be added to a regression by making each category option a dummy variable and including them all (minus one)

$$Y_i = \beta_0 + \beta_2 D_1 + \beta_2 D_2 + \beta_3 D_3$$

    where observations can fall into category 1, 2, 3, or 4
  - Including all category option dummies into a regression yields the **dummy variable trap**, where all dummies are perfectly multicollinear
  - Must drop one category dummy, the "reference group"
  - Coefficients on dummy variables are the difference between that category and the reference category:
    - $\beta_0 = Y$ for category 4 (omitted)
    - $\beta_1 =$ difference between category 1 and category 4 (omitted)
    - $\beta_2 =$ difference between category 2 and category 4 (omitted)
    - $\beta_3 =$ difference between category 3 and category 4 (omitted)
- **Interaction terms** measure if there is an additional effect of one variable on the value of another, 3 combinations:

  1. Between a dummy and a continuous variable

$$Y_i = \beta_0 + \beta_1 X_i + \beta_2 D_i + \beta_3 \mathbf{X_i} \times \mathbf{D_i}$$

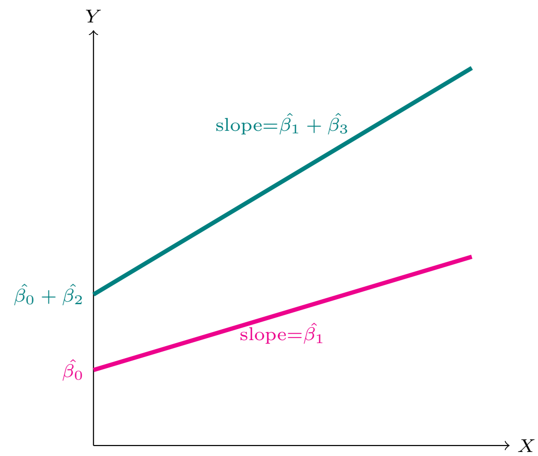
     - Coefficients:
       - β_0: Y_i for $X_i = 0$ and $D_i = 0$
       - β_1: Effect of $X_i \rightarrow Y_i$ for $D_i = 0$
       - β_2: Effect on Y_i of difference between $D_i = 0$ and $D_i = 1$

        * β_3: Effect of *difference* of $X_i \rightarrow Y_i$ between $D_i = 0$ and $D_i = 1$
    - Easier to see as two different regression lines:
        * When $D_i = 0$ (Control group):

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$$

        * When $D_i = 1$ (Treatment group):

$$\hat{Y}_i = (\hat{\beta}_0 + \hat{\beta}_2) + (\hat{\beta}_1 + \hat{\beta}_3) X_i$$



        * Two regression lines may have (same/different) intercepts and (same/different) intercepts, test significance of:
            · β_2: difference in intercepts
            · β_3: difference in slopes

2. Between two dummy variables

$$Y_i = \beta_0 + \beta_1 D_{1i} + \beta_2 D_{2i} + \beta_3 \mathbf{D_{1i}} \times \mathbf{D_{2i}}$$

    - Coefficients:
        * β_0: value of Y for $D_{1i} = 0$ and $D_{2i} = 0$
        * β_1: effect on Y of $D_{1i} = 0 \rightarrow 1$ when $D_{2i} = 0$
        * β_2: effect on Y of $D_{2i} = 0 \rightarrow 1$ when $D_{1i} = 0$
        * β_3: *increment* to effect on Y of $D_{1i} = 0 \rightarrow 1$ when $D_{2i} = 1$ vs. when $D_{2i} = 0$
    - Compare difference in group means:
        * $D_{1i} = 0, D_{2i} = 0$: $\widehat{Y}_i = \hat{\beta}_0$
        * $D_{1i} = 0, D_{2i} = 1$: $\widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_2$
        * $D_{1i} = 1, D_{2i} = 0$: $\widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1$
        * $D_{1i} = 1, D_{2i} = 1$: $\widehat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3$

3. Between two continuous variables

$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i} + \beta_3 (\mathbf{X_{1i}} \times \mathbf{X_{2i}})$$

    - Marginal effects:
        * $\dfrac{\Delta Y_i}{\Delta X_{1i}} = \beta_1 + \beta_3 X_{2i}$ — marginal effect of $X_{1i} \rightarrow Y_i$ depends on X_{2i}
        * $\dfrac{\Delta Y_i}{\Delta X_{2i}} = \beta_2 + \beta_3 X_{1i}$ — marginal effect of $X_{2i} \rightarrow Y_i$ depends on X_{1i}

Transforming Variables

- Polynomial functions
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{\beta}_2 X_i^2 + ... + \hat{\beta}_r X_i^r + \epsilon_i$$
where r is highest power X_i is raised to, a function with $r-1$ bends
  - Quadratic model
$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{\beta}_2 X_i^2 + \epsilon_i$$
    * Marginal effect of $X_i \rightarrow Y_i$:
$$\frac{d\,Y_i}{d\,X_i} = \hat{\beta}_1 + 2\hat{\beta}_2 X_i$$
    * Value of X_i where Y_i is minimized/maximized:
$$X_i^* = -\frac{1}{2}\frac{\beta_1}{\beta_2}$$
  - To determine if a higher-powered term is necessary, test significance of its associated coefficient (e.g. β_2 for quadratic model above)
  - To determine if a model is nonlinear, run F-test of all higher-powered terms
- Logarithmic functions (ln)
  - Natural Logs (ln) are used to talk about percentage changes, 3 types of models:
    1. Linear-log model:
$$Y = \beta_0 + \beta_1 \mathbf{ln}(\mathbf{X})$$
       * β_1: A 1% change in $X \rightarrow \frac{\beta_1}{100}$ unit change in Y
    2. Log-linear model:
$$\mathbf{ln}(\mathbf{Y}) = \beta_0 + \beta_1 X$$
       * β_1: A 1 unit change in $X \rightarrow 100 \times \beta_1\%$ change in Y
    3. Log-log model:
$$\mathbf{ln}(\mathbf{Y}) = \beta_0 + \beta_1 \mathbf{ln}(\mathbf{X})$$
       * β_1: A 1% change in $X \rightarrow \beta_1\%$ change in Y (**elasticity** between X and Y)
- Standardized coefficients
$$Y_i = \beta_0 + \beta_1 X_{1i} + \beta_2 X_{2i}$$
  - To compare the magnitude of marginal effects (e.g. is $\beta_1 > \beta_2$) across variables of different units, standardize the variables by taking the Z-score of all observations
$$Variable_{std} = \frac{Variable - \overline{Variable}}{sd(Variable)}$$
  - Coefficients measure the # of standard deviations change of Y a 1 std. dev. change in X causes
- Joint Hypothesis Testing
  - Joint hypothesis tests against the null hypothesis of a value for multiple parameters, e.g.
$$H_0 : \beta_1 = 0, \beta_2 = 0$$
$$H_1 : H_0 \text{ is false}$$
  - Three common tests
    1. H_0: $\beta_1 = \beta_2 = 0$, testing if multiple variables do not affect Y

2. H_0: $\beta_1 = \beta_2$, testing if multiple variables have the same effect (must be same units)
3. H_0: all β's$= 0$, the model overall explains no variation in Y

- In general, with q restrictions:

$$H_0 : \beta_j = \beta_{j,0}, \beta_k = \beta_{k,0}, \text{...for } q \text{ restrictions}$$

- Use the F-statistic, (simplified homoskedastic formula below)

$$F_{q,n-k-1} = \frac{\left(\frac{(R_u^2 - R_r^2)}{q}\right)}{\left(\frac{(1 - R_u^2)}{(n-k-1)}\right)}$$

- Compares the R^2's of two models:
  * Unrestricted model: regression with all coefficients
  * Restricted model: regression under the null hypothesis (e.g. where $\beta_1 = 0, \beta_2 = 0$)
- F tests if the increase in R^2 from including the suspect variables ($Restricted \rightarrow Unrestricted$) increases by a statistically significant amount

Panel Data

- Panel data tracks the same individuals (a cross-section) over time (time-series)

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \epsilon_{it}$$

  with N number of i groups and T number of t time periods

- A **pooled model** simply runs this as normal OLS regression
  - Biased: ignores factors correlated with X in ϵ
  - Systematic differences across groups i that may be stable over time
  - Systematic differences across time t that may be stable across groups

- (One-Way) Fixed effects model

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \alpha_{\mathbf{i}} + \nu_{it}$$

  - α_i: group-fixed effect (pulled from error term ϵ_{it})
    * Includes *all* differences across groups that do not change over time! (e.g. geography, culture, etc. of Maryland vs. Alaska)
    * Does *not* include variables that change over time!
    * Estimates a different intercept for each group
  - Least Squares Dummy Variable (LSDV) Approach: can estimate via creating & including a dummy variable for each group (minus 1 to avoid dummy variable trap)

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \sum_{i=1}^{N-1} \alpha_i D_i$$

    where α_i is a coefficient and D_i is a dummy variable for group i, for example:

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \beta_2 Alabama_i + \beta_3 Alaska_i + ...$$

- Two-Way Fixed effects model

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \alpha_{\mathbf{i}} + \tau_{\mathbf{t}} + \nu_{it}$$

  - τ_i: time-fixed effect (pulled from error term ϵ_{it})
    * Includes *all* differences over time that do not change across groups! (e.g. all States experience recession in 2008, or federal law change)
    * Does *not* include variables that are different across groups!
    * Estimates a different intercept for each time period
  - Least Squares Dummy Variable (LSDV) Approach: can estimate via creating & including a dummy variable for each group and each time period (minus 1 for each to avoid dummy variable trap)

$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \sum_{i=1}^{N-1} \alpha_i D_i + \sum_{t=1}^{T-1} \tau_i D_t$$

    where α_i and τ_t are coefficients, D_i is a dummy variable for group i, and D_t is a dummy variable for time period t, for example:

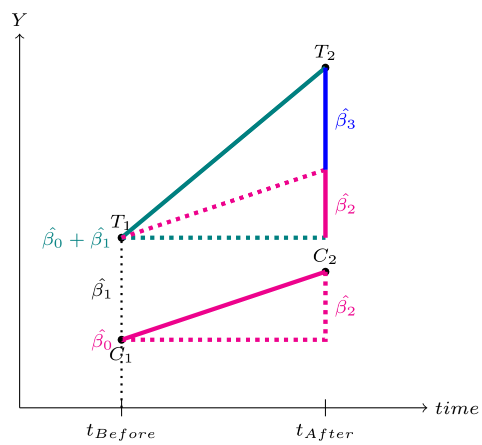
$$\widehat{Y_{it}} = \beta_0 + \beta_1 X_{it} + \beta_2 Alabama_i + \beta_3 Alaska_i + ... + \beta_{51} 2000_t + \beta_{52} 2001_t + ...$$

- Difference-in-Differences model

$$\widehat{Y_{it}} = \beta_0 + \beta_1 \text{Treated}_i + \beta_2 \text{After}_{it} + \beta_3 (\text{Treated}_i \times \text{After}_t) + \epsilon_{it}$$

  - Where:
    - $\text{Treated}_i = 1$ if unit i is in treatment group
    - $\text{After}_{it} = 1$ if observation it is after treatment period

	Control	Treatment	Group Diff. (ΔY_i)
Before	β_0	$\beta_0 + \beta_1$	β_1
After	$\beta_0 + \beta_2$	$\beta_0 + \beta_1 + \beta_2 + \beta_3$	$\beta_1 + \beta_3$
Time Diff. (ΔY_t)	β_2	$\beta_2 + \beta_3$	β_3
			Diff-in-diff ($\Delta\Delta Y$)



$$\Delta\Delta Y = (Treated_{after} - Treated_{before}) - (Control_{after} - Control_{before})$$

  - OLS Coefficients:
    - $\hat{\beta}_0$: value of Y for control before treatment
    - $\hat{\beta}_1$: difference between treatment and control (before treatment)
    - $\hat{\beta}_2$: time difference between before and after treatment
    - $\hat{\beta}_3$: difference-in-difference: effect of treatment
  - Values of Y for different groups:
    - Y for Control Group Before: $\hat{\beta}_0$
    - Y for Control Group After: $\hat{\beta}_0 + \hat{\beta}_2$
    - Y for Treatment Group Before: $\hat{\beta}_0 + \hat{\beta}_1$
    - Y for Treatment Group After: $\hat{\beta}_0 + \hat{\beta}_1 + \hat{\beta}_2 + \hat{\beta}_3$
    - Treatment Effect: $\hat{\beta}_3$
  - Key assumption about *counterfactual*: if not for treatment, the treated group would change the same over time as the control group (parallel time trends, magenta dotted line)
  - Can generalize the model with two way fixed effects:

$$\widehat{Y_{it}} = \alpha_i + \tau_t + \beta_3 (\text{Treated}_i \times \text{After}_t) + X_{it} + \epsilon_{it}$$

    - α_i: group-fixed effects, where some groups receive treatment and others do not
    - τ_t: time-fixed effects, where some periods are before treatment and others are after
    - X_{it}: other control variables
    - This allows for multiple treatments to happen at different times!