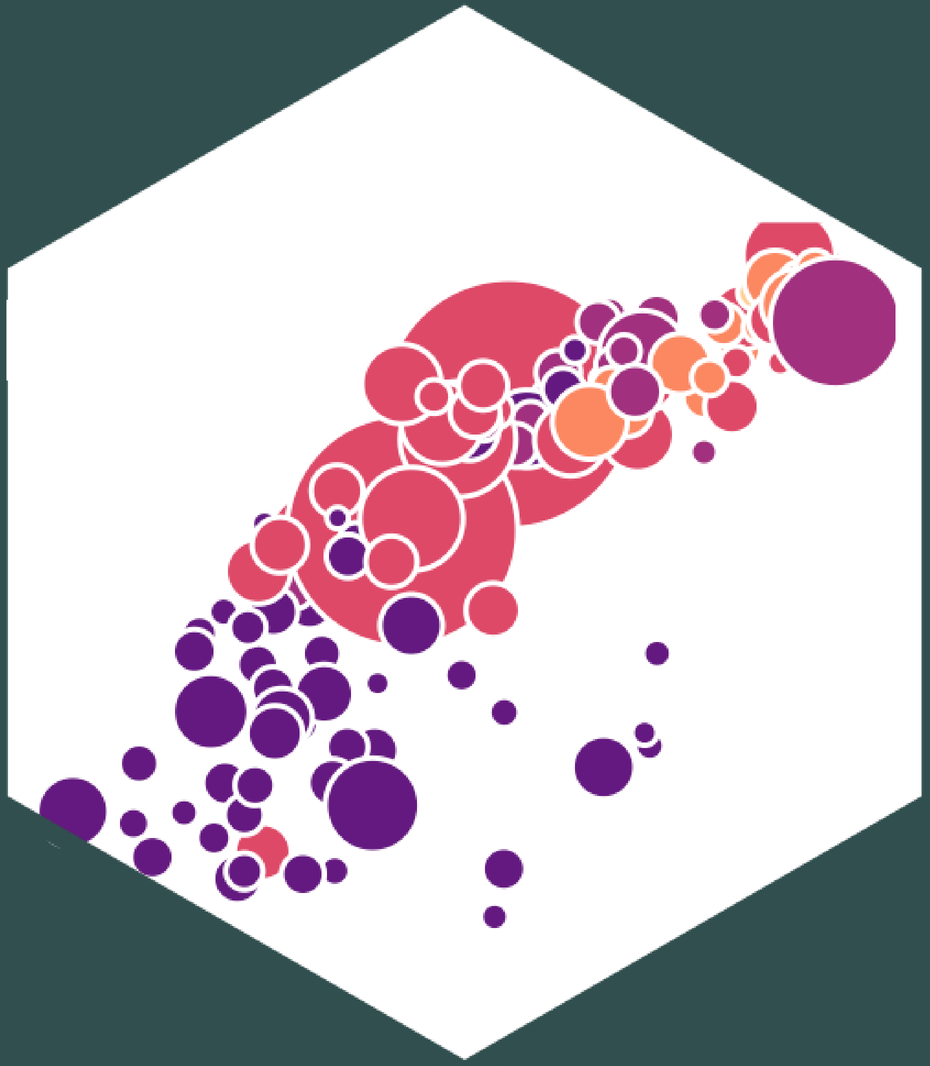
4.3 — Nonlinearity & Transformatio



ECON 480 • Econometrics • Fall 2022

Dr. Ryan Safner

Associate Professor of Economics

safner@hood.edu

ryansafner/metricsF22

metricsF22.classes.ryansafner.com

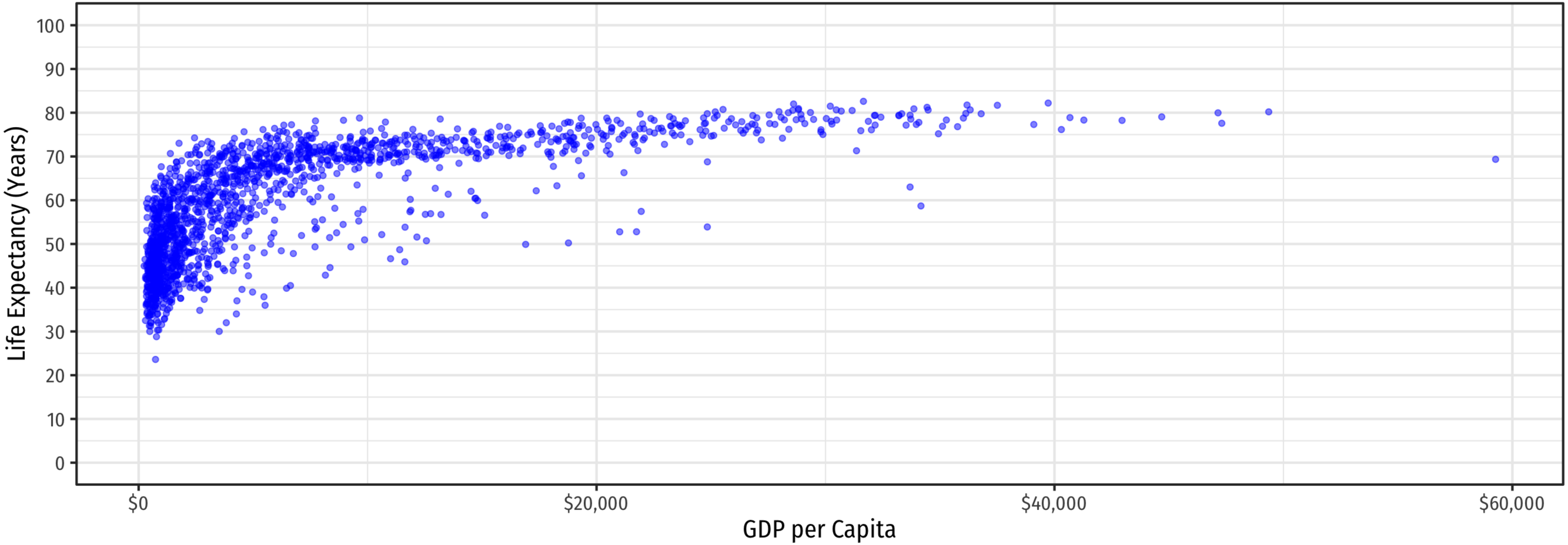
Contents

Nonlinear Effects

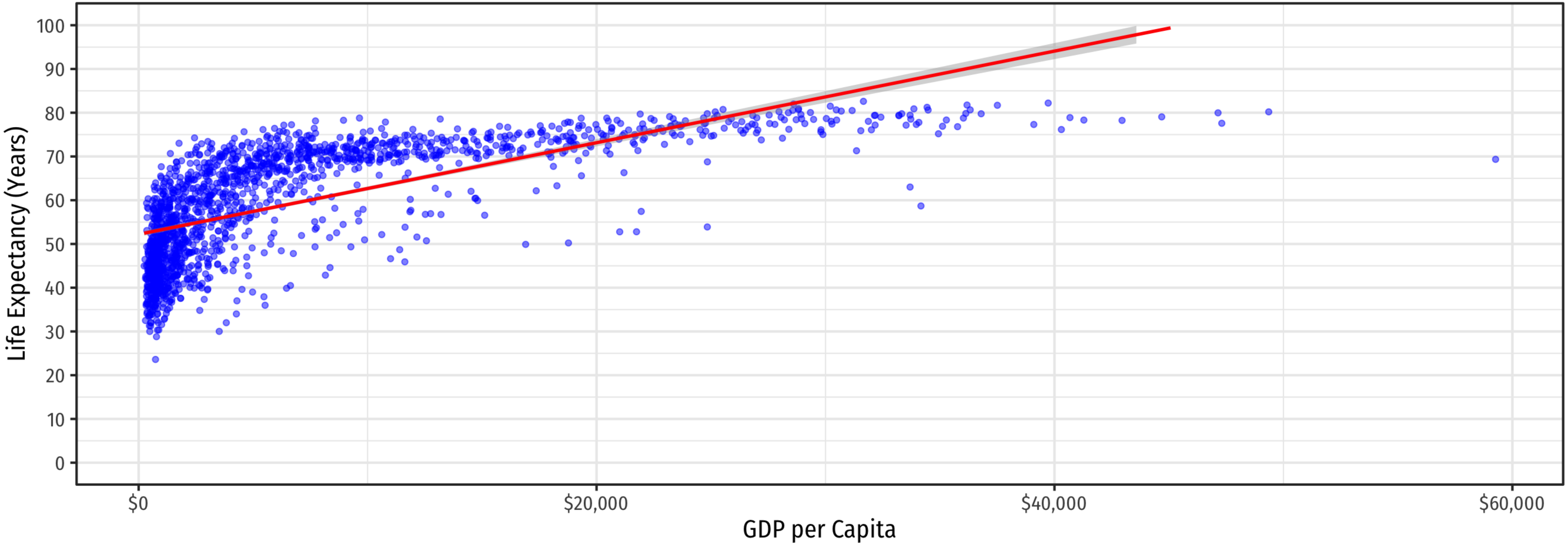
Linear Regression

- OLS is commonly known as "***linear regression***" as it fits a **straight line** to data points
- Often, data and relationships between variables may *not* be linear

Linear Regression

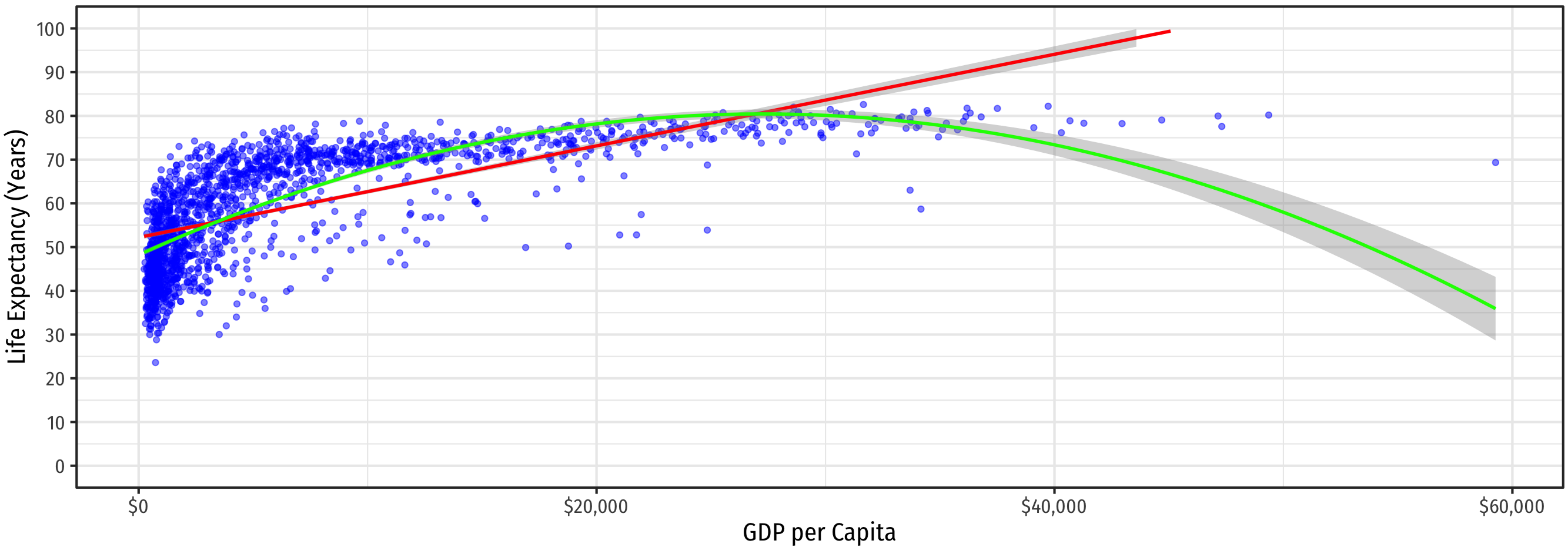


Linear Regression



$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i$$

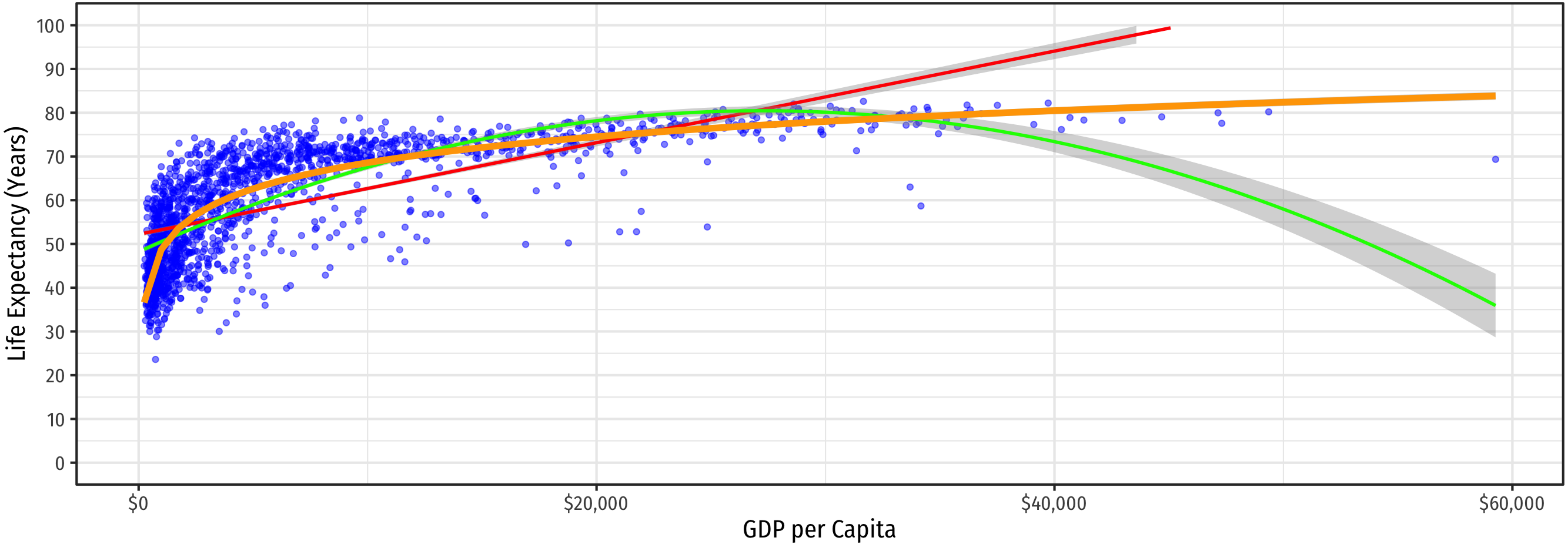
Linear Regression



$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i$$

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i + \hat{\beta}_2 \text{GDP}_i^2$$

Linear Regression



$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i$$

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i + \hat{\beta}_2 \text{GDP}_i^2$$

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \ln \text{GDP}_i$$

Sources of Nonlinearities

- Effect of $X_1 \rightarrow Y$ might be nonlinear if:

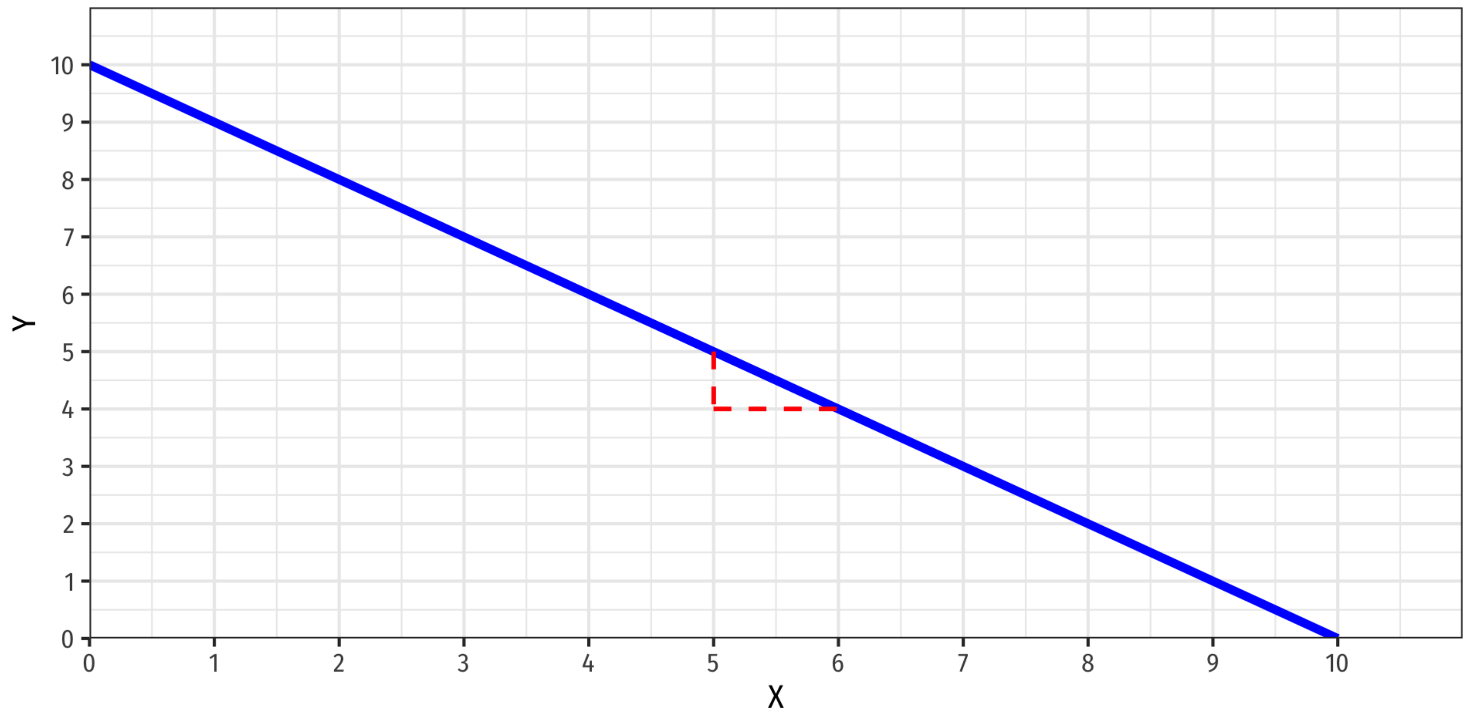
1. $X_1 \rightarrow Y$ is different for different levels of X_1
   - e.g. **diminishing returns**: $\uparrow X_1$ increases Y at a *decreasing* rate
   - e.g. **increasing returns**: $\uparrow X_1$ increases Y at an *increasing* rate
2. $X_1 \rightarrow Y$ is different for different levels of X_2
   - e.g. interaction effects (last lesson)

Nonlinearities Alter Marginal Effects

- **Linear**:

$$Y = \hat{\beta}_0 + \hat{\beta}_1 X$$

- marginal effect (slope), $(\hat{\beta}_1) = \frac{\Delta Y}{\Delta X}$ is constant for all X

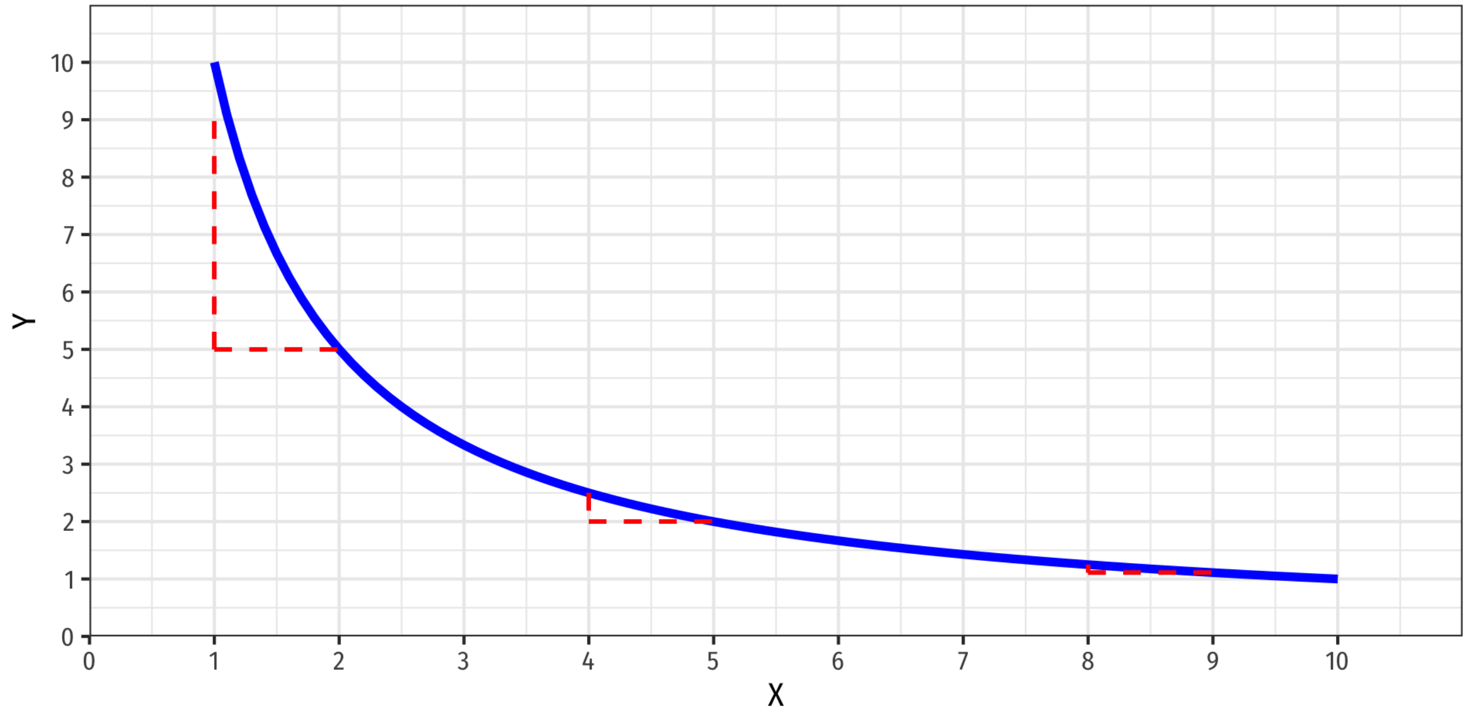


Nonlinearities Alter Marginal Effects

- **Polynomial**:

$$Y = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$

- Marginal effect, "slope" $\left(\neq \hat{\beta}_1\right)$ *depends on the value of* X!

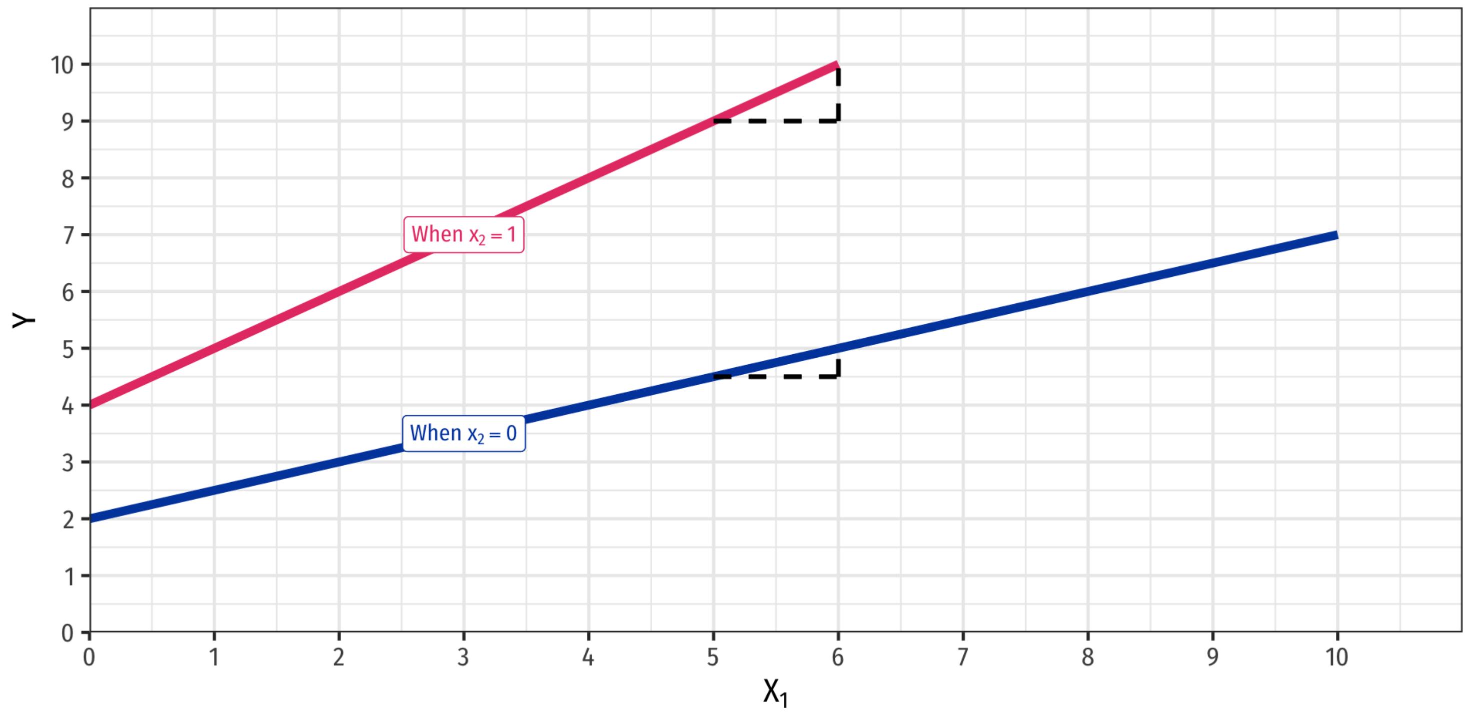


Nonlinearities Alter Marginal Effects

- **Interaction Effect**:

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X_1 + \hat{\beta}_2 X_2 + \hat{\beta}_3 X_1 \times X_2$$

- Marginal effect, "slope" *depends on the value of* X_2!
- Easy example: if X_2 is a dummy variable:
  - $X_2 = 0$ (control) vs. $X_2 = 1$ (treatment)

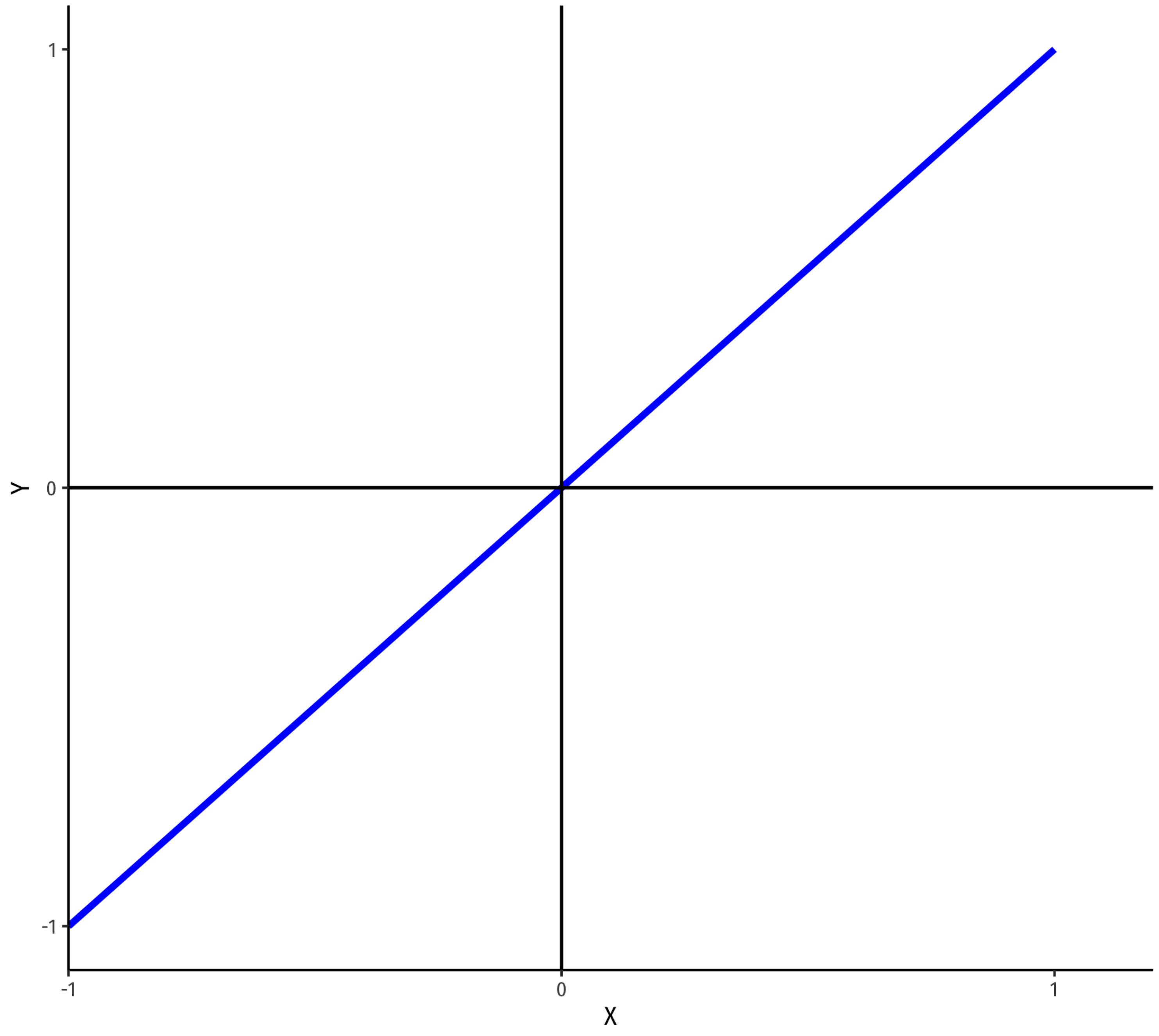


Polynomial Models

Polynomial Functions of X I

- Linear

$$\hat{Y} = \hat{\beta_0} + \hat{\beta_1} X$$



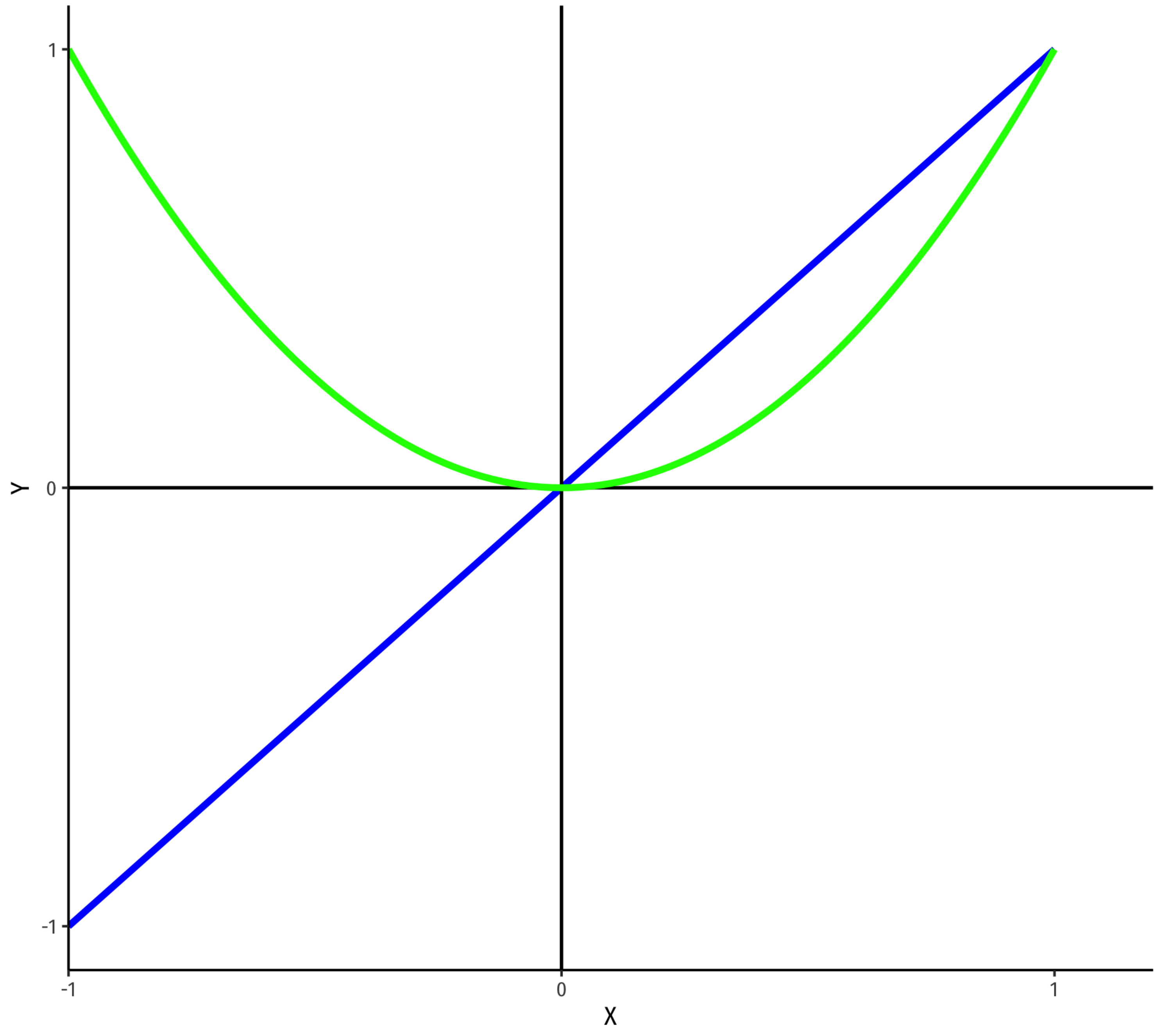
Polynomial Functions of X I

- Linear

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

- Quadratic

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$



Polynomial Functions of X I

- Linear

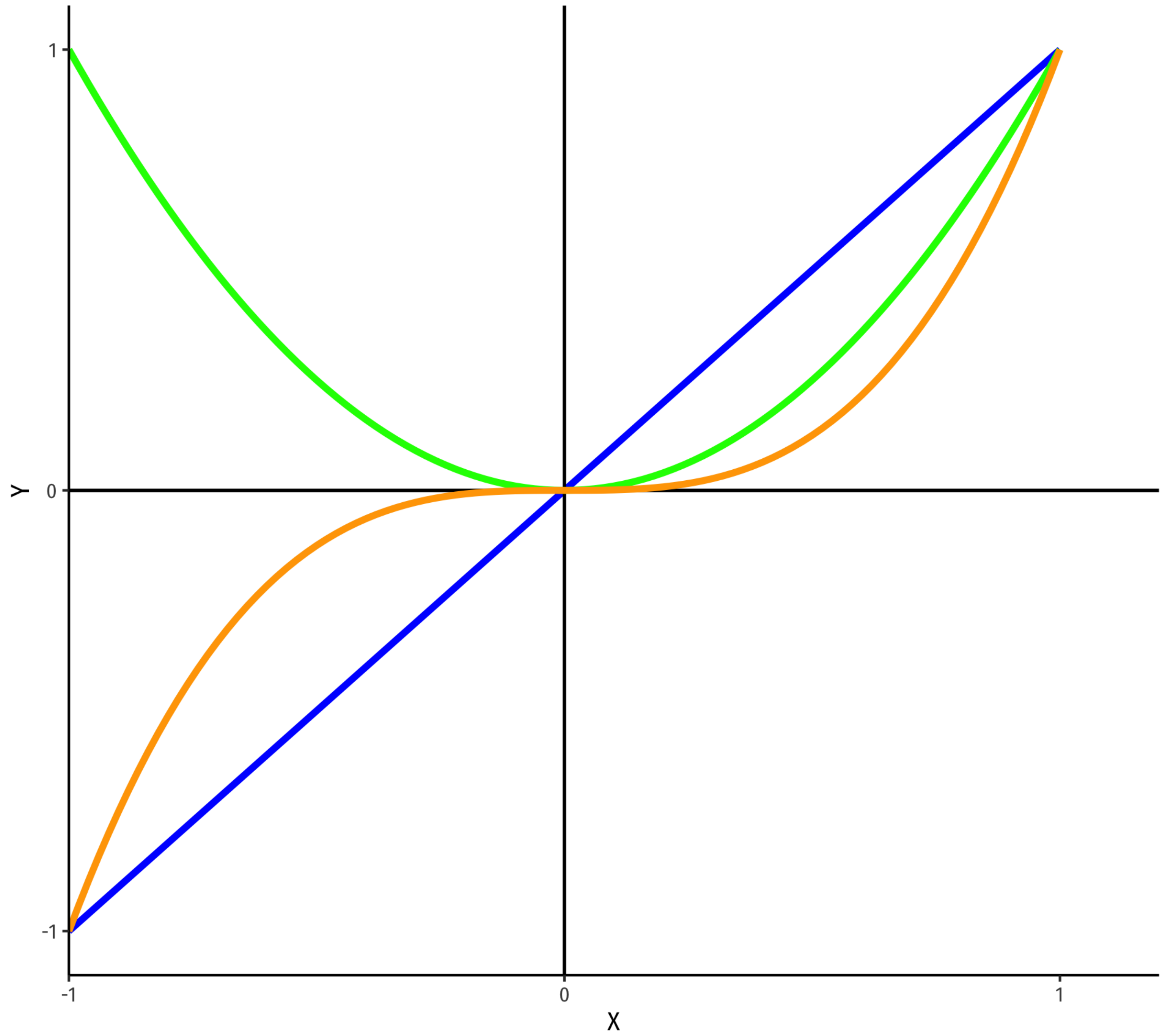
$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

- Quadratic

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$

- Cubic

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2 + \hat{\beta}_3 X^3$$



Polynomial Functions of X I

- Linear

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$

- Quadratic

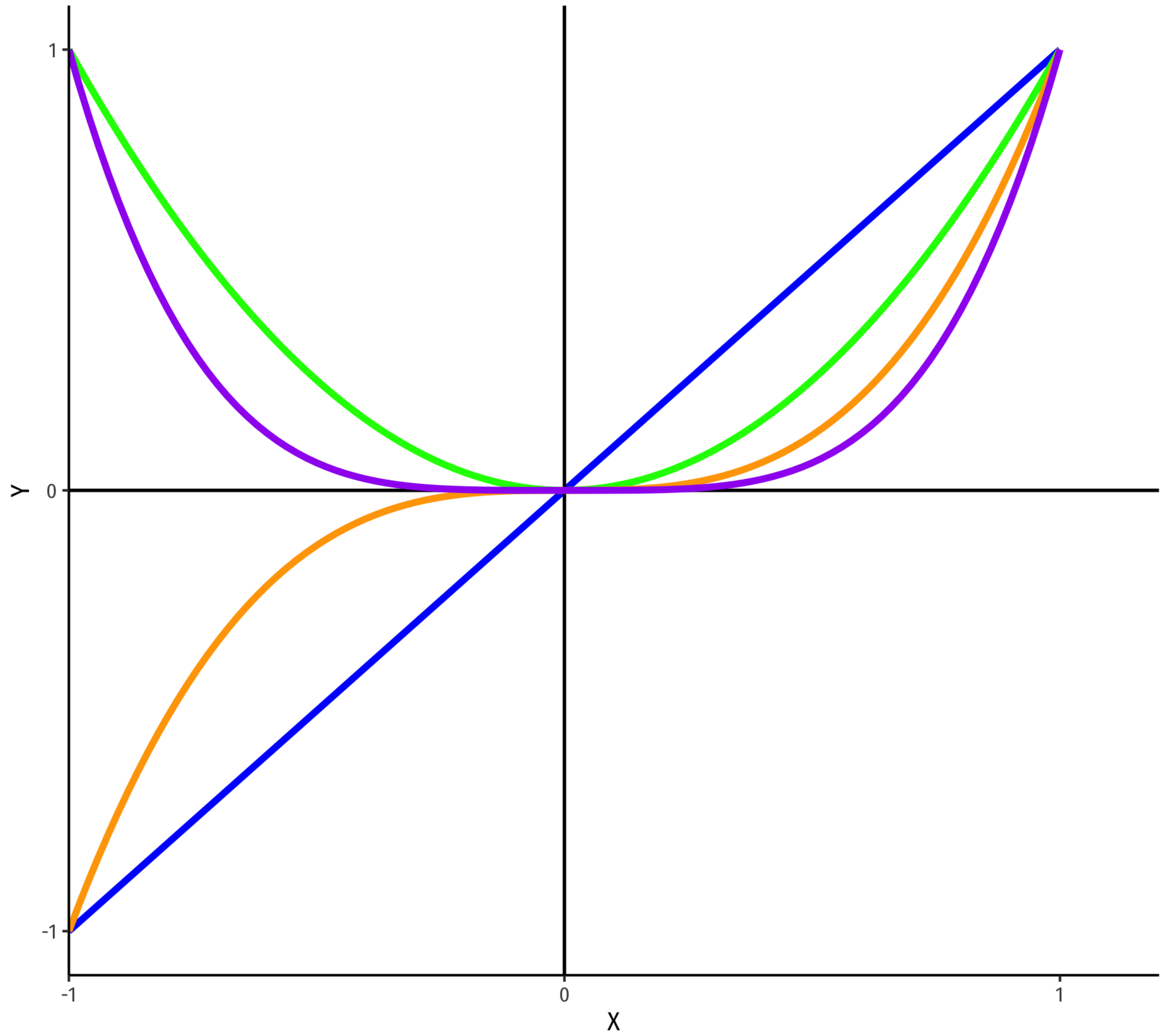
$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2$$

- Cubic

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2 + \hat{\beta}_3 X^3$$

- Quartic

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X + \hat{\beta}_2 X^2 + \hat{\beta}_3 X^3 + \hat{\beta}_4 X^4$$



Polynomial Functions of X II

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{\beta}_2 X_i^2 + \cdots + \hat{\beta}_r X_i^r + u_i$$

- Where r is the highest power X_i is raised to
  - quadratic $r = 2$
  - cubic $r = 3$
- The graph of an r^{th}-degree polynomial function has $(r-1)$ bends
- Just another multivariate OLS regression model!

Quadratic Model

Quadratic Model

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{\beta}_2 X_i^2$$

- **Quadratic model** has X and X^2 variables in it (yes, need both!)
- How to interpret coefficients (betas)?
  - β_0 as “intercept” and β_1 as “slope” makes no sense 🧐
  - β_1 as effect $X_i \rightarrow Y_i$ holding X_i^2 constant??[1]
- **Estimate marginal effects** by calculating predicted $\hat{Y}_i$ for different levels of X_i

1. Note: this is *not* a perfect multicollinearity problem! Correlation only measures *linear* relationships!

Quadratic Model: Calculating Marginal Effects

$$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 X_i + \hat{\beta}_2 X_i^2$$

- What is the **marginal effect** of $\Delta X_i \rightarrow \Delta Y_i$?
- Take the **derivative** of Y_i with respect to X_i:

$$\frac{\partial Y_i}{\partial X_i} = \hat{\beta}_1 + 2\hat{\beta}_2 X_i$$

- **Marginal effect** of a 1 unit change in X_i is a $\left(\hat{\beta}_1 + 2\hat{\beta}_2 X_i\right)$ unit change in Y

Quadratic Model: Example I

Example

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ GDP per capita}_i + \hat{\beta}_2 \text{ GDP per capita}_i^2$$

- Use `gapminder` package and data

```r
library(gapminder)
```

Quadratic Model: Example II

- These coefficients will be very large, so let's transform `gdpPercap` to be in $1,000's

```
gapminder <- gapminder %>%
  mutate(GDP_t = gdpPercap/1000)

gapminder %>% head() # look at it
```

country <fct>	continent <fct>	year <int>
Afghanistan	Asia	1952
Afghanistan	Asia	1957
Afghanistan	Asia	1962
Afghanistan	Asia	1967
Afghanistan	Asia	1972
Afghanistan	Asia	1977

6 rows | 1-3 of 7 columns

Quadratic Model: Example II

- Let's also create a squared term, `gdp_sq`

```
gapminder <- gapminder %>%
  mutate(GDP_sq = GDP_t^2)

gapminder %>% head() # look at it
```

country <fct>	continent <fct>	year <int>
Afghanistan	Asia	1952
Afghanistan	Asia	1957
Afghanistan	Asia	1962
Afghanistan	Asia	1967
Afghanistan	Asia	1972
Afghanistan	Asia	1977

6 rows | 1-3 of 8 columns

Quadratic Model: Example IV

- Can "manually" run a multivariate regression with `GDP_t` and `GDP_sq`

```r
library(broom)
reg1 <- lm(lifeExp ~ GDP_t + GDP_sq, data = gapminder)

reg1 %>% tidy()
```

term <chr>	estimate <dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
GDP_sq	-0.01501927

3 rows | 1-2 of 5 columns

Quadratic Model: Example IV

- OR use `gdp_t` and add the `I()` operator to transform the variable in the regression, `I(gdp_t^2)`[1]

```r
reg1_alt <- lm(lifeExp ~ GDP_t + I(GDP_t^2), data = gapminder)

reg1_alt %>% tidy()
```

term	estimate
<chr>	<dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
I(GDP_t^2)	-0.01501927

3 rows | 1-2 of 5 columns

1. Here is a decent explanation of what `I()` does. An alternative is to use `poly(GDP_t, 2)` to make the squared term, but this has some issues

Quadratic Model: Example V

term <chr>	estimate <dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
GDP_sq	-0.01501927

3 rows | 1-2 of 5 columns

$$\widehat{\text{Life Expectancy}}_i = 50.52 + 1.55\,\text{GDP}_i - 0.02\,\text{GDP}_i^2$$

- Positive effect ($\hat{\beta}_1 > 0$), with diminishing returns ($\hat{\beta}_2 < 0$)
- Marginal effect of GDP on Life Expectancy **depends on initial value of GDP!**

Quadratic Model: Example VI

term <chr>	estimate <dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
GDP_sq	-0.01501927

3 rows | 1-2 of 5 columns

- **Marginal effect** of GDP on Life Expectancy:

$$\frac{\partial Y}{\partial X} = \hat{\beta}_1 + 2\hat{\beta}_2 X_i$$

$$\frac{\partial \text{Life Expectancy}}{\partial \text{GDP}} \approx 1.55 + 2(-0.02)\,\text{GDP}$$

$$\approx 1.55 - 0.04\,\text{GDP}$$

Quadratic Model: Example VII

$$\frac{\partial \text{Life Expectancy}}{\partial \text{GDP}} = 1.55 - 0.04\,\text{GDP}$$

Marginal effect of GDP if GDP $= 5$ ($ thousand):

$$\begin{aligned}\frac{\partial \text{Life Expectancy}}{\partial \text{GDP}} &= 1.55 - 0.04\text{GDP}\\ &= 1.55 - 0.04(5)\\ &= 1.55 - 0.20\\ &= 1.35\end{aligned}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy increases by 1.35 years

Quadratic Model: Example VIII

$$\frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} = 1.55 - 0.04 \, \text{GDP}$$

Marginal effect of GDP if GDP $= 25$ ($ thousand):

$$\begin{aligned} \frac{\partial \, \text{Life Expectancy}}{\partial \, \text{GDP}} &= 1.55 - 0.04\text{GDP} \\ &= 1.55 - 0.04(25) \\ &= 1.55 - 1.00 \\ &= 0.55 \end{aligned}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy increases by 0.55 years

Quadratic Model: Example X

$$\frac{\partial \text{Life Expectancy}}{\partial \text{GDP}} = 1.55 - 0.04\,\text{GDP}$$

Marginal effect of GDP if GDP $= 50$ ($ thousand):

$$\begin{aligned}\frac{\partial \text{Life Expectancy}}{\partial \text{GDP}} &= 1.55 - 0.04\text{GDP} \\ &= 1.55 - 0.04(50) \\ &= 1.55 - 2.00 \\ &= -0.45\end{aligned}$$

- i.e. for every addition $1 (thousand) in GDP per capita, average life expectancy *decreases* by 0.45 years

Quadratic Model: Example XI

$$\widehat{\text{Life Expectancy}}_i = 50.52 + 1.55\,\text{GDP per capita}_i - 0.02\,\text{GDP per capita}_i^2$$

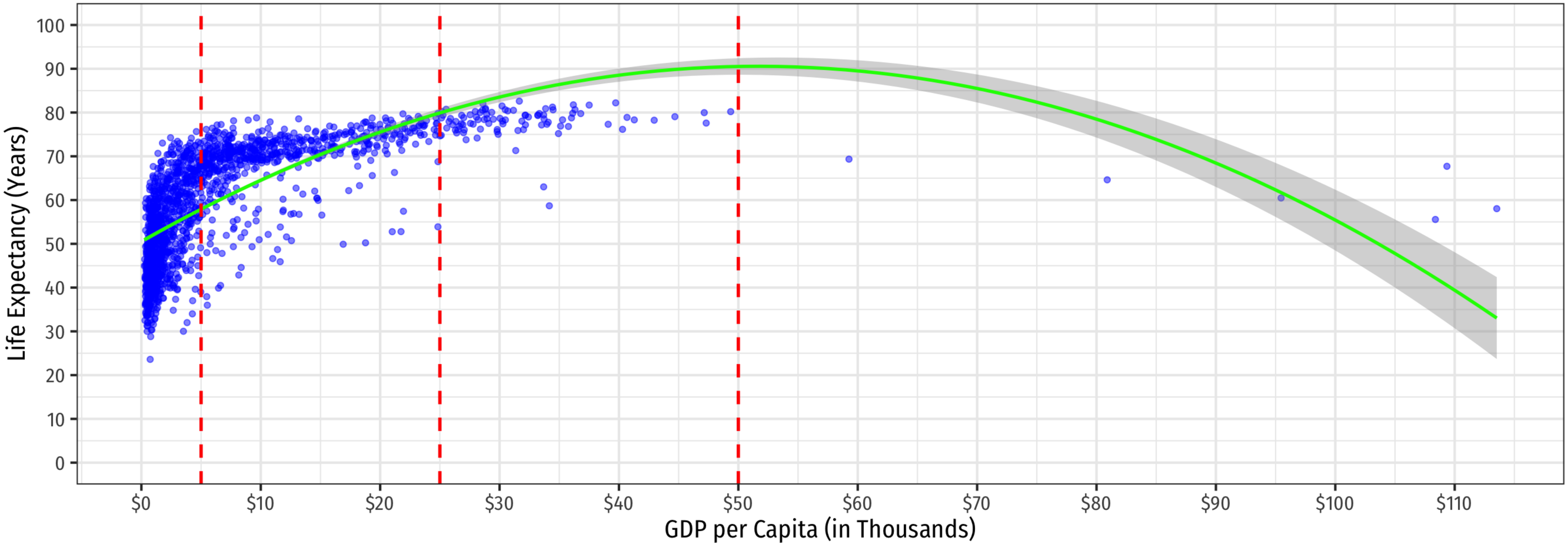
$$\frac{\partial\,\text{Life Expectancy}}{d\,\text{GDP}} = 1.55 - 0.04\text{GDP}$$

***Initial* GDP per capita**	**Marginal Effect[1]**
\$5,000	1.35 years
\$25,000	0.55 years
\$50,000	-0.45 years

1. Of +\$1,000 GDP/capita on Life Expectancy

Quadratic Model: Example XII

- Code



Quadratic Model: Maxima and Minima I

- For a polynomial model, we can also find the predicted **maximum** or **minimum** of $\hat{Y}_i$
- A quadratic model has a single global maximum or minimum (1 bend)
- By calculus, a minimum or maximum occurs where:

$$\begin{aligned} \frac{\partial Y_i}{\partial X_i} &= 0 \\ \beta_1 + 2\beta_2 X_i &= 0 \\ 2\beta_2 X_i &= -\beta_1 \\ X_i^* &= -\frac{\beta_1}{2\beta_2} \end{aligned}$$

Quadratic Model: Maxima and Minima II

term <chr>	estimate <dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
GDP_sq	-0.01501927

3 rows | 1-2 of 5 columns

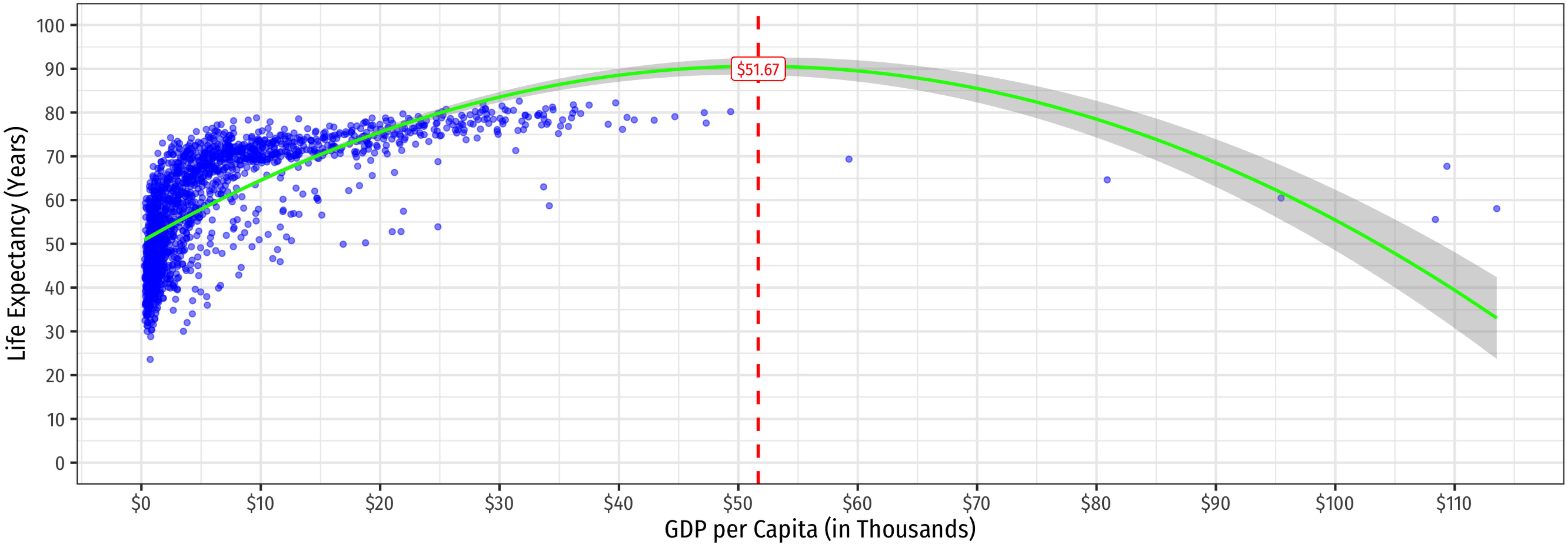
$$GDP_i^* = -\frac{\beta_1}{2\beta_2}$$

$$GDP_i^* = -\frac{(1.55)}{2(-0.015)}$$

$$GDP_i^* \approx 51.67$$

Quadratic Model: Maxima and Minima III

- Code



Determining If Polynomials Are Necessary I

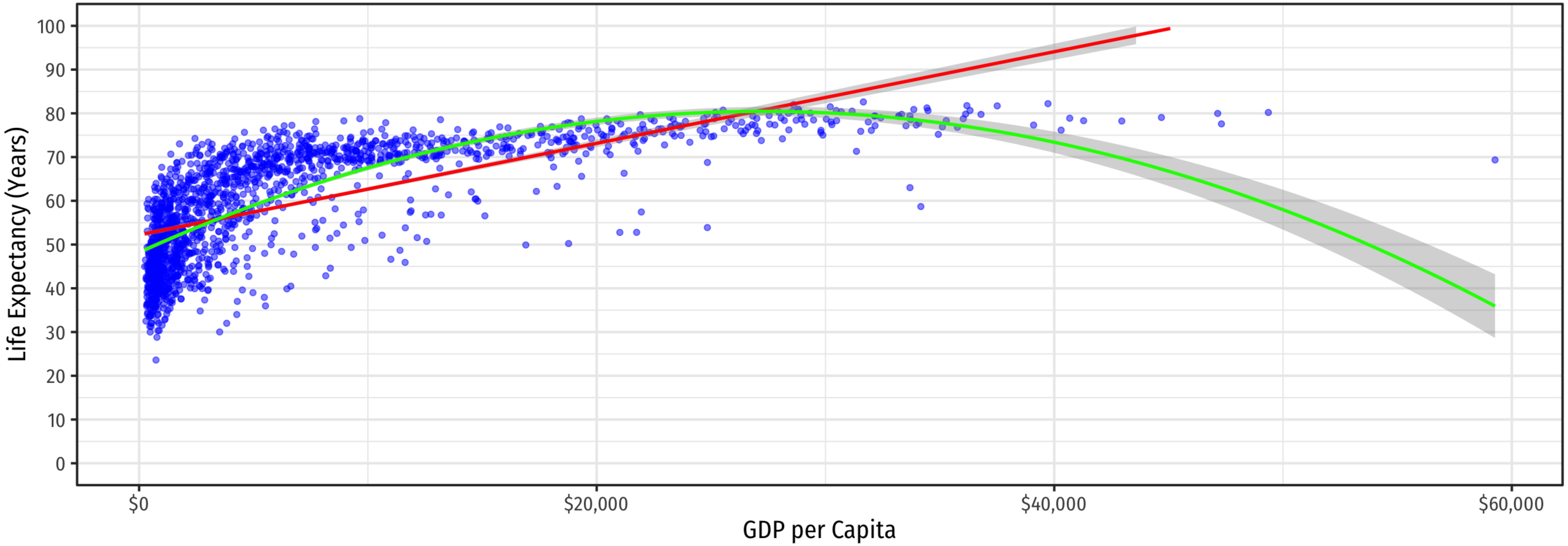
term <chr>	estimate <dbl>
(Intercept)	50.52400578
GDP_t	1.55099112
GDP_sq	-0.01501927

3 rows | 1-2 of 5 columns

- Is the quadratic term necessary?
- Determine if $\hat{\beta}_2$ (on X_i^2) is statistically significant:
  - $H_0: \hat{\beta}_2 = 0$
  - $H_a: \hat{\beta}_2 \neq 0$

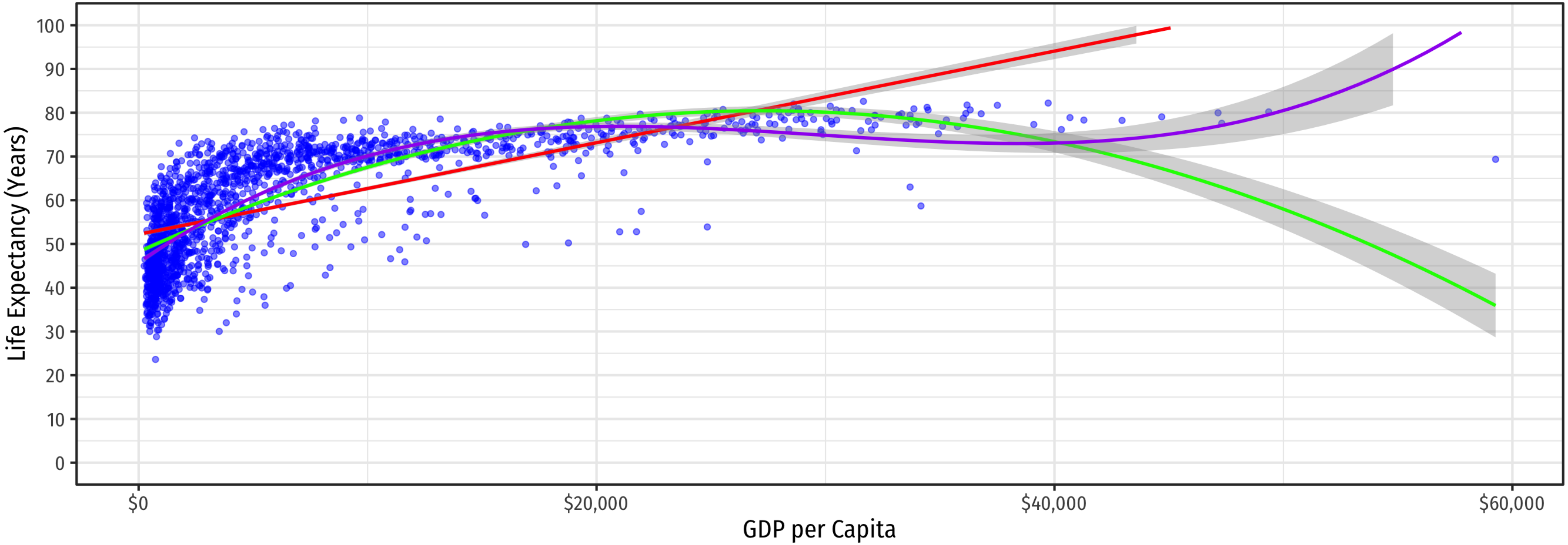
Determining Polynomials are Necessary II

- Should we keep going up in polynomials?



Determining Polynomials are Necessary II

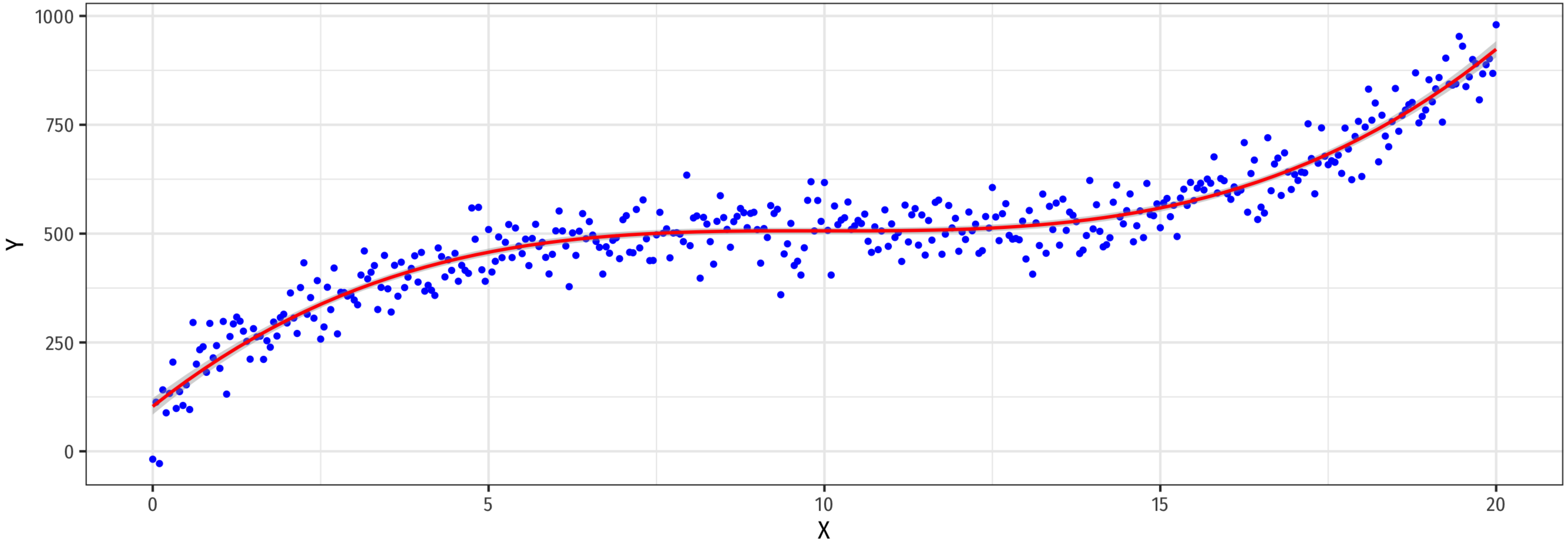
- Should we keep going up in polynomials?



$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 GDP_i + \hat{\beta}_2 GDP_i^2 + \hat{\beta}_3 GDP_i^3$$

Determining Polynomials are Necessary III

- In general, you should have a **compelling theoretical reason** why data or relationships should **"change direction"** multiple times
- Or clear data patterns that have multiple "bends"
- Recall, we care more about accurately measuring the causal effect of $X \rightarrow Y$, rather than getting the most accurate prediction possible for $\hat{Y}$



Determining Polynomials are Necessary IV

term <chr>	estimate <dbl>
(Intercept)	47.4755069510
GDP_t	2.7226370698
I(GDP_t^2)	-0.0681545071
I(GDP_t^3)	0.0004093149

4 rows | 1-2 of 5 columns

- $\hat{\beta}_3$ is statistically significant…
- …but can we really think of a good reason to complicate the model?

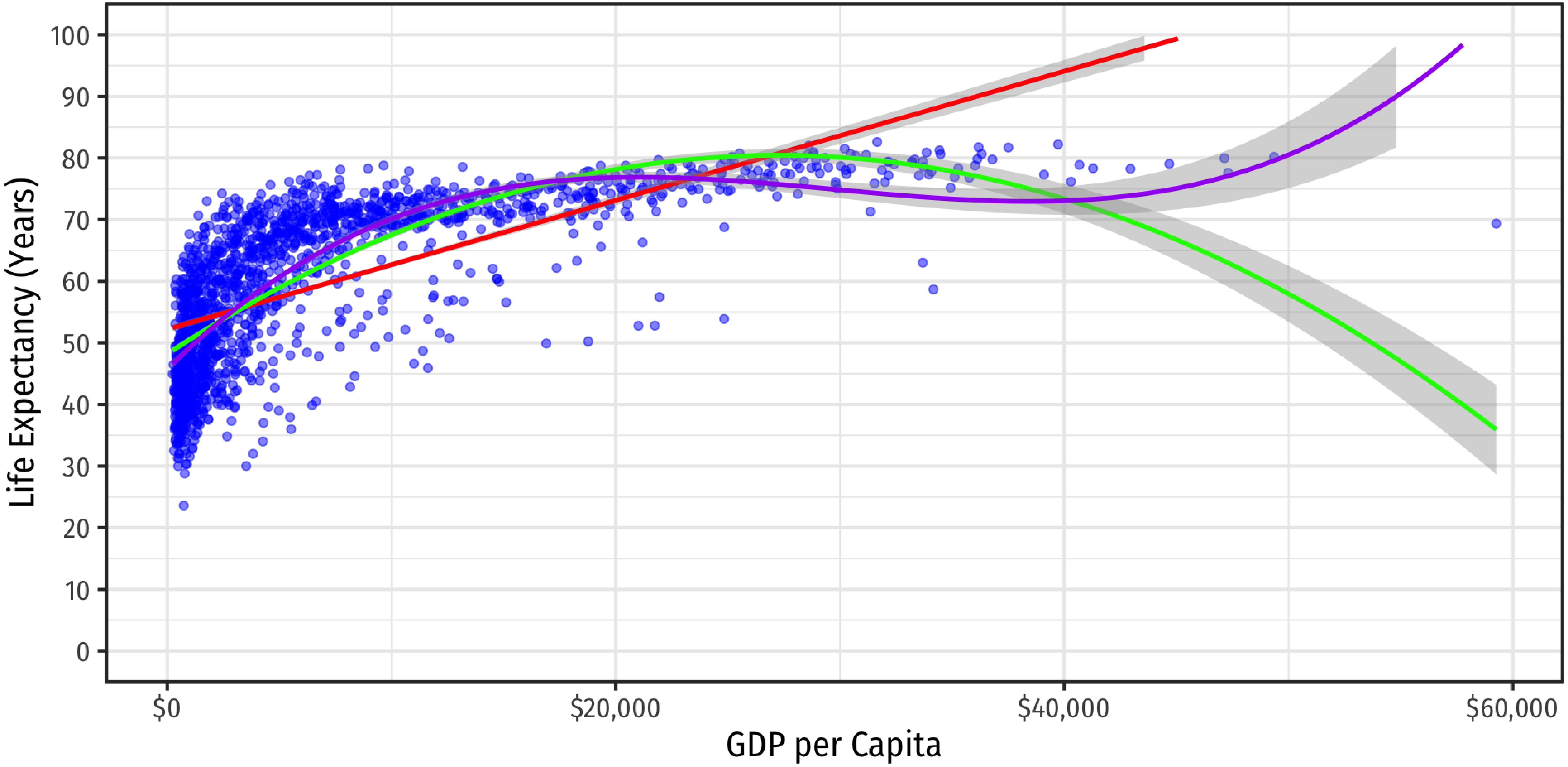
If You Kept Going...

term <chr>	estimate <dbl>	std.error <dbl>
(Intercept)	4.003294e+01	5.846282e-01
GDP_t	8.722968e+00	5.290582e-01
I(GDP_t^2)	-1.081312e+00	1.294759e-01
I(GDP_t^3)	7.190930e-02	1.334295e-02
I(GDP_t^4)	-2.705563e-03	7.010624e-04
I(GDP_t^5)	6.063170e-05	2.056983e-05
I(GDP_t^6)	-8.254873e-07	3.495442e-07
I(GDP_t^7)	6.685309e-09	3.408241e-09
I(GDP_t^8)	-2.956581e-11	1.766287e-11
I(GDP_t^9)	5.490732e-14	3.765889e-14

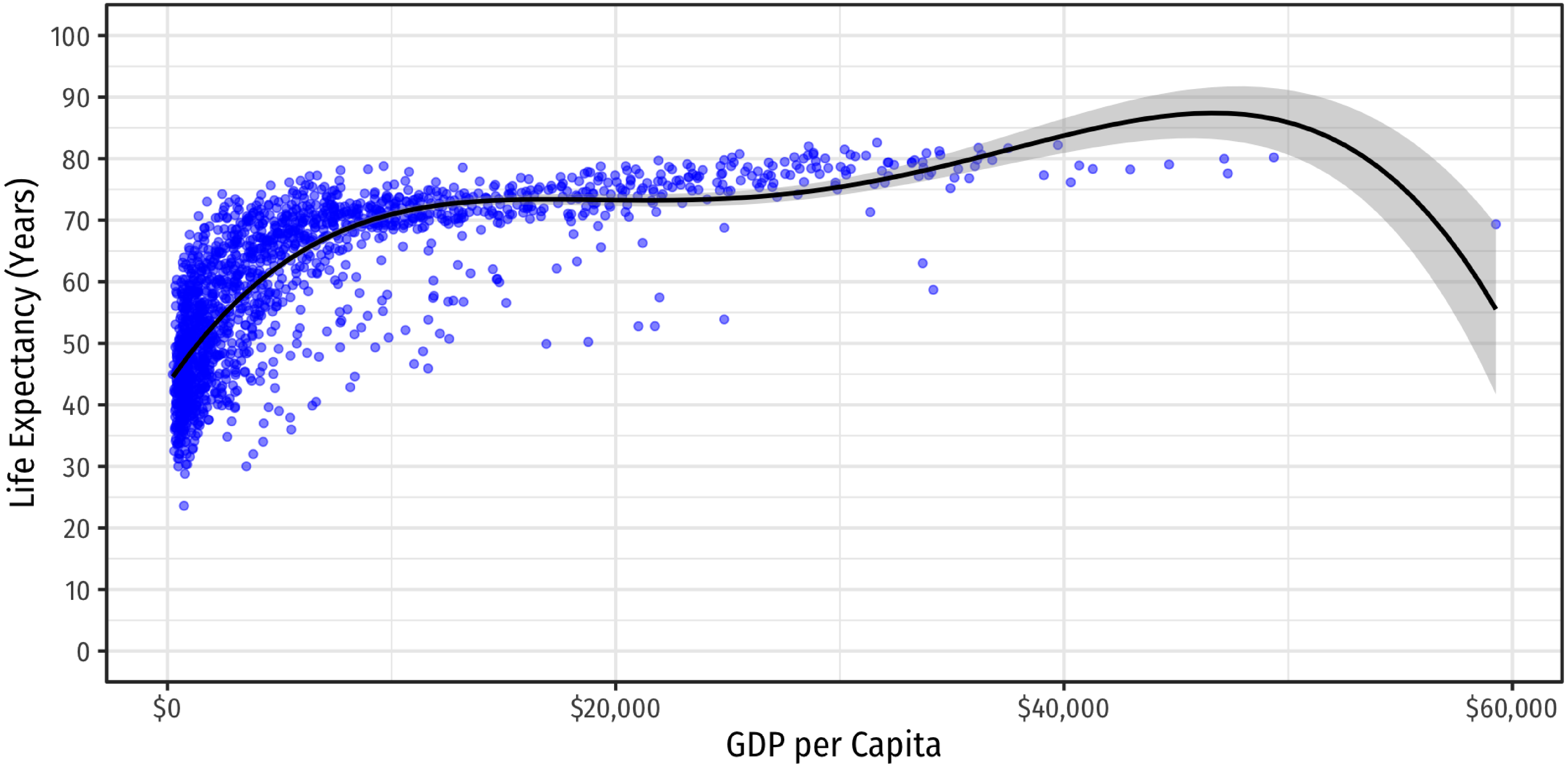
1-10 of 10 rows | 1-3 of 5 columns

- It takes until a 9th-degree polynomial for one of the terms to become insignificant...
- ...but does this make the model *better*? *more interpretable*?
- A famous problem of **overfitting**

If You Kept Going...Visually

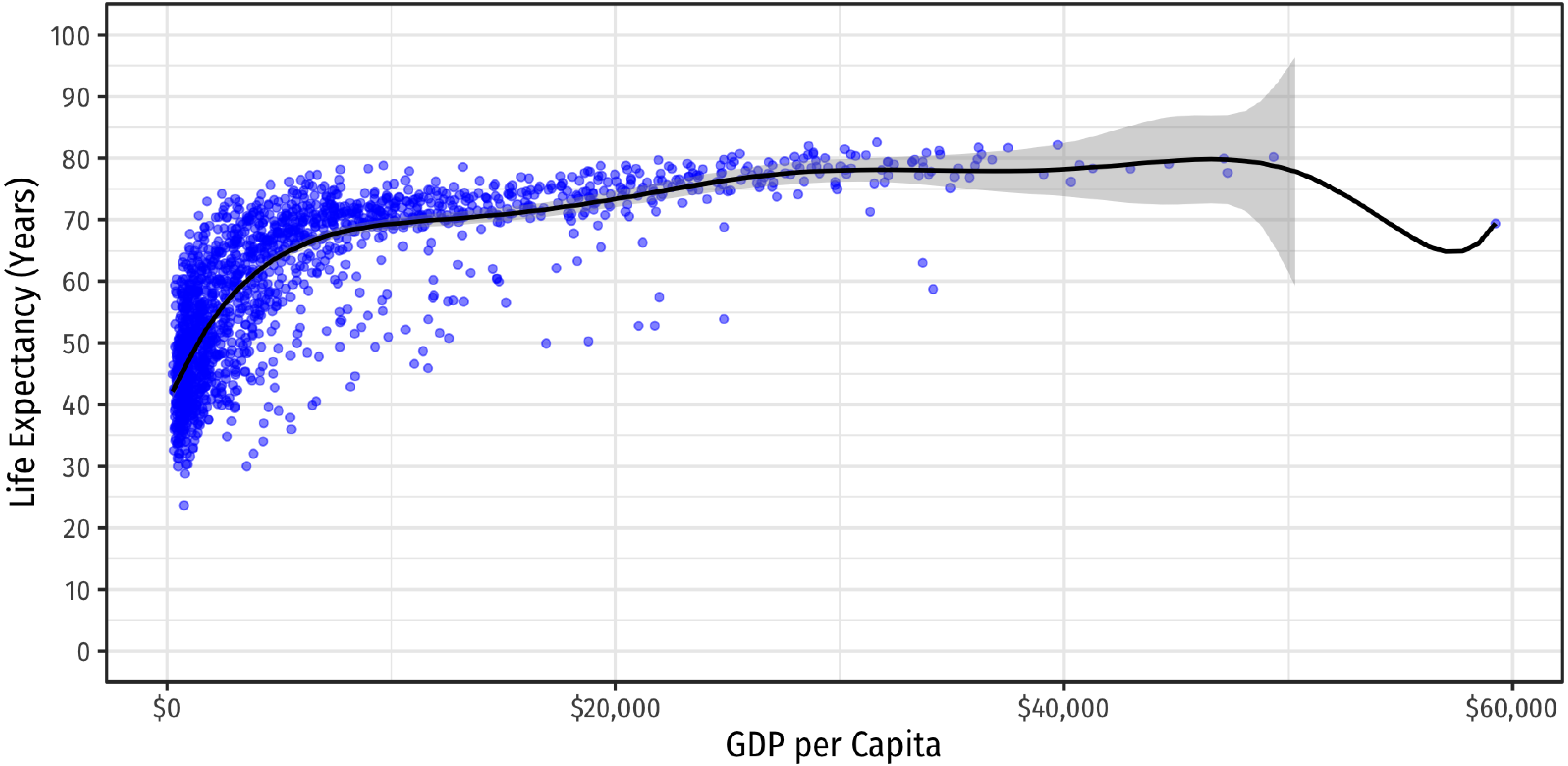


If You Kept Going...Visually



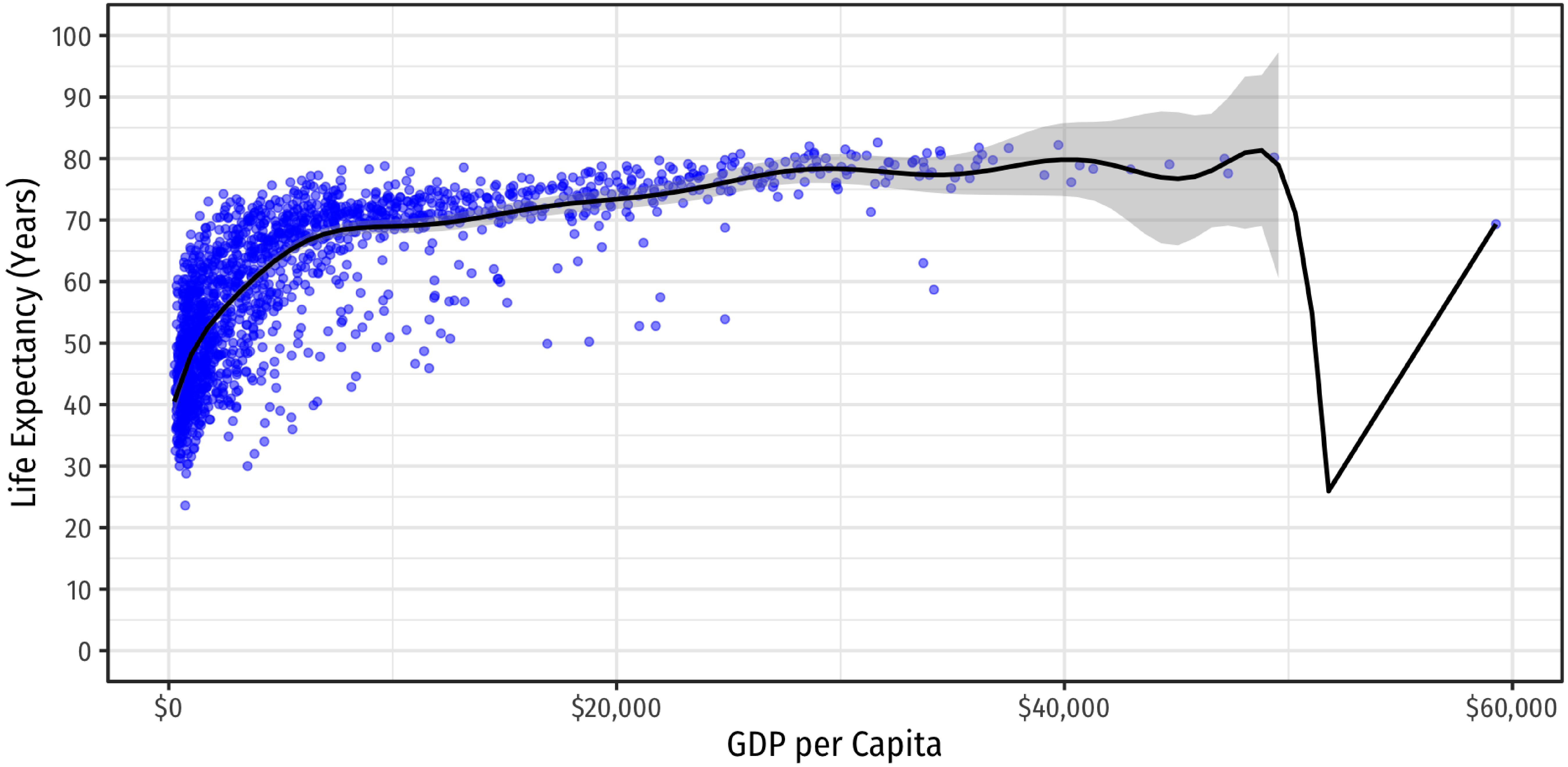
A 4th-degree polynomial

If You Kept Going...Visually



A 9th-degree polynomial

If You Kept Going...Visually



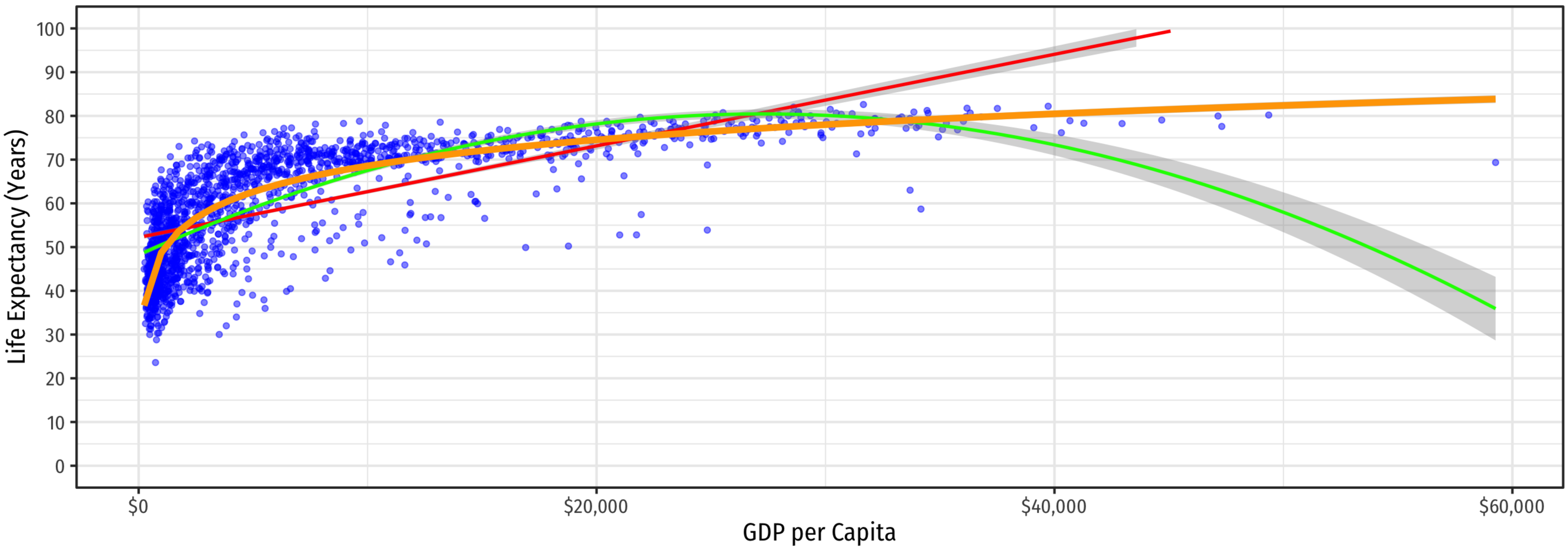
A 14th-degree polynomial

Strategy for Polynomial Model Specification

1. Are there good theoretical reasons for relationships changing (e.g. increasing/decreasing returns)?
2. Plot your data: does a straight line fit well enough?
3. Specify a polynomial function of a higher power (start with 2) and estimate OLS regression
4. Use t-test to determine if higher-power term is significant
5. Interpret effect of change in X on Y
6. Repeat steps 3-5 as necessary (if there are good theoretical reasons)

Logarithmic Models

Linear Regression



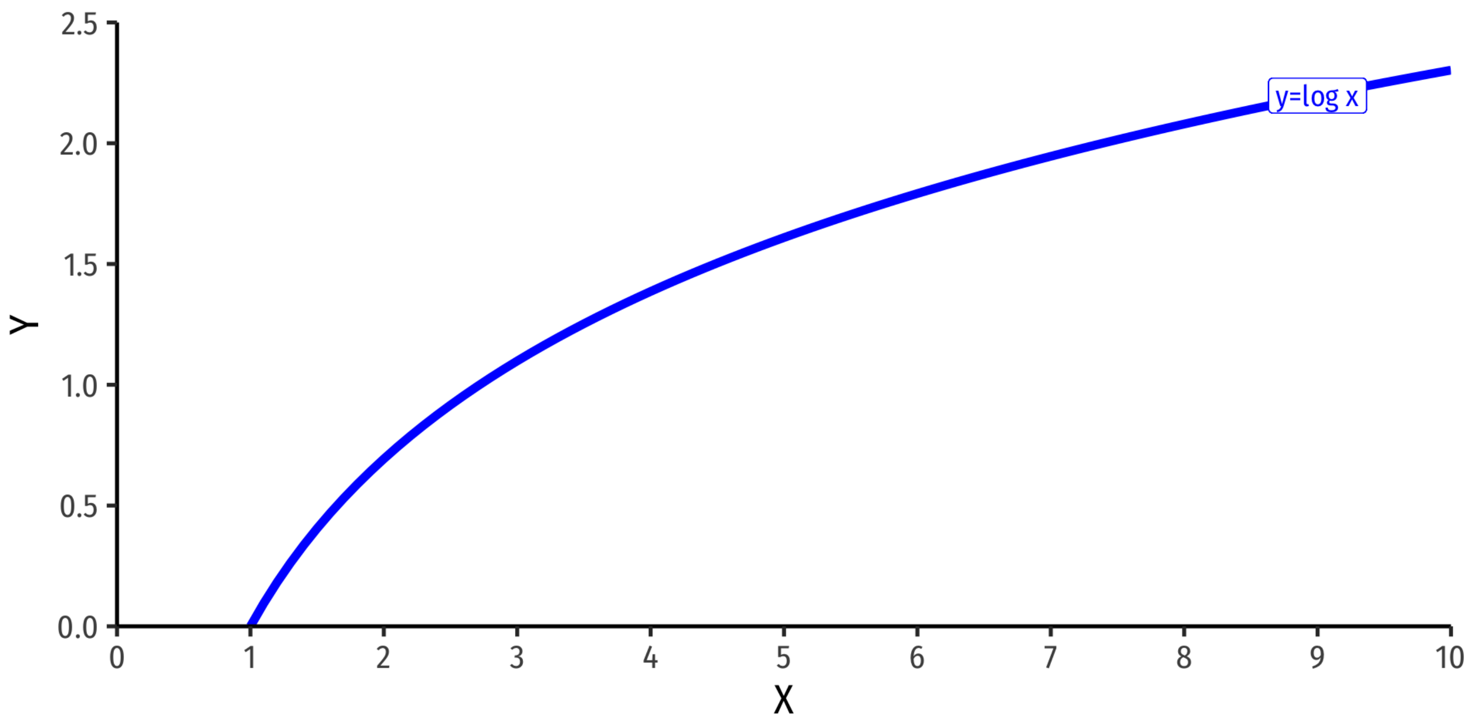
$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i$$

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{GDP}_i + \hat{\beta}_2 \text{GDP}_i^2$$

$$\widehat{\text{Life Expectancy}}_i = \hat{\beta}_0 + \hat{\beta}_1 \ln \text{GDP}_i$$

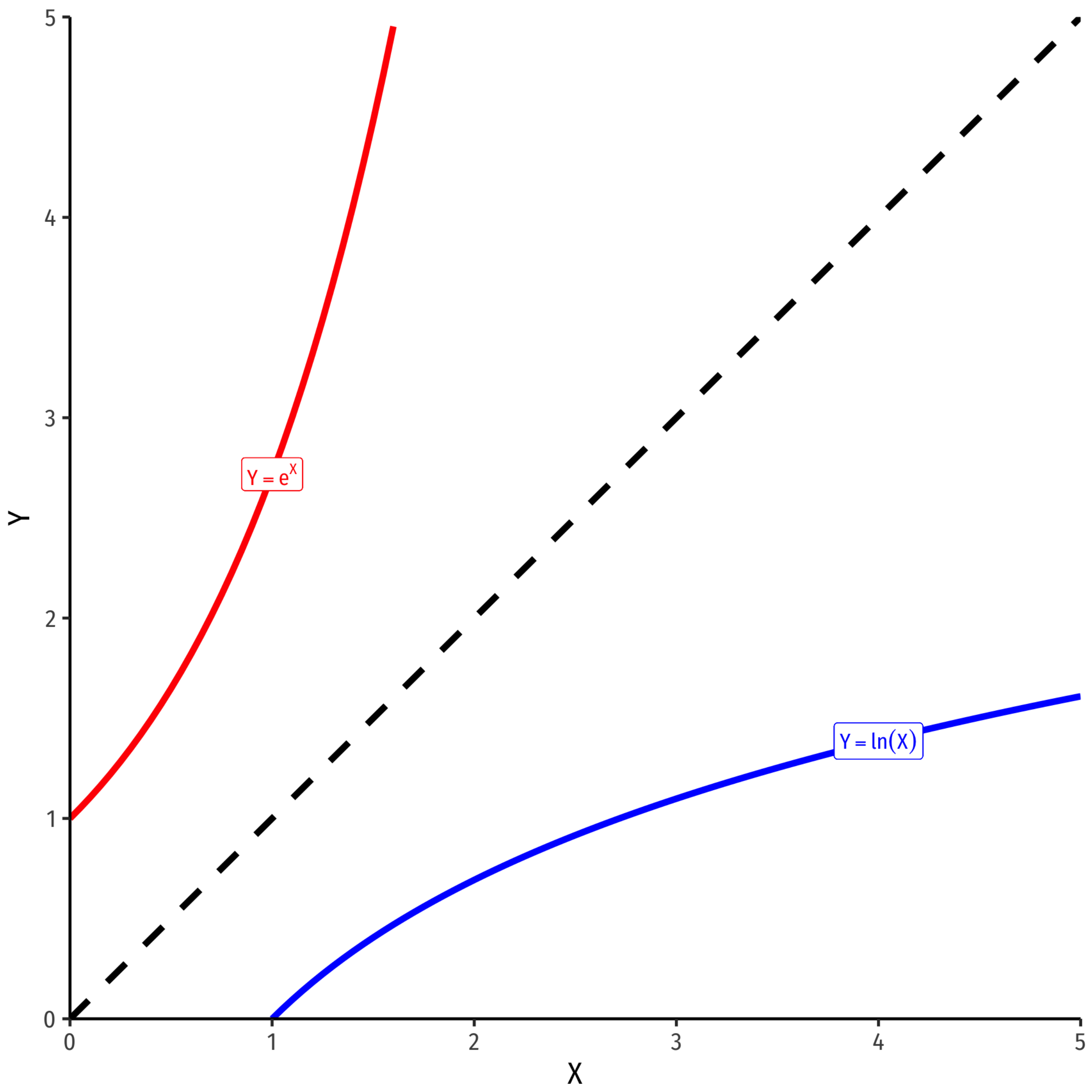
Logarithmic Models

- Another useful model for nonlinear data is the **logarithmic model**[1]
  - We transform either X, Y, or *both* by taking the **(natural) logarithm**
- Logarithmic model has two additional advantages
  1. We can easily interpret coefficients as **percentage changes** or **elasticities**
  2. Useful economic shape: diminishing returns (production functions, utility functions, etc)



1. Don't confuse this with a **logistic (logit) model** for *dependent* dummy variables.

The Natural Logarithm



- The exponential function, $Y = e^X$ or $Y = exp(X)$, where base $e = 2.71828...$
- Natural logarithm is the inverse, $Y = ln(X)$

The Natural Logarithm: Review I

- **Exponents** are defined as

$$b^n = \underbrace{b \times b \times \cdots \times b}_{n \text{ times}}$$

- where base b is multiplied by itself n times
- **Example**: $2^3 = \underbrace{2 \times 2 \times 2}_{n=3} = 8$
- **Logarithms** are the inverse, defined as the exponents in the expressions above

$$\text{If } b^n = y \text{, then } log_b(y) = n$$

- n is the number you must raise b to in order to get y
- **Example**: $log_2(8) = 3$

The Natural Logarithm: Review II

- Logarithms can have any base, but common to use the **natural logarithm** $(\ln)$ with base $\mathbf{e = 2.71828...}$

$$\text{If } e^n = y, \text{ then } \ln(y) = n$$

The Natural Logarithm: Properties

- Natural logs have a lot of useful properties:

1. $\ln(\frac{1}{x}) = -\ln(x)$
2. $\ln(ab) = \ln(a) + \ln(b)$
3. $\ln(\frac{x}{a}) = \ln(x) - \ln(a)$
4. $\ln(x^a) = a \ln(x)$
5. $\frac{d \ln x}{d x} = \frac{1}{x}$

The Natural Logarithm: Example

- Most useful property: for small change in x, Δx:

$$\underbrace{\ln(x+\Delta x) - \ln(x)}_{\text{Difference in logs}} \approx \underbrace{\frac{\Delta x}{x}}_{\text{Relative change}}$$

Example

Let $x = 100$ and $\Delta x = 1$, relative change is:

$$\frac{\Delta x}{x} = \frac{(101-100)}{100} = 0.01 \text{ or } 1\%$$

- The logged difference:

$$\ln(101) - \ln(100) = 0.00995 \approx 1\%$$

- This allows us to very easily interpret coefficients as **percent changes** or **elasticities**

Elasticity

- An **elasticity** between any two variables, $\epsilon_{Y,X}$ describes the **responsiveness** (in %) of one variable (Y) to a change in another (X)

$$\epsilon_{Y,X} = \frac{\%\Delta Y}{\%\Delta X} = \frac{\left(\frac{\Delta Y}{Y}\right)}{\left(\frac{\Delta X}{X}\right)}$$

- Numerator is relative change in Y, Denominator is relative change in X

- **Interpretation**: a 1% change in X will cause a $\epsilon_{Y,X}$% change in Y

Math FYI: Cobb Douglas Functions and Logs

- One of the (many) reasons why economists love Cobb-Douglas functions:

$$Y = AL^{\alpha}K^{\beta}$$

- Taking logs, relationship becomes linear:

$$\ln(Y) = \ln(A) + \alpha \ln(L) + \beta \ln(K)$$

- With data on (Y, L, K) and linear regression, can estimate α and β
  - α: elasticity of Y with respect to L
    - A 1% change in L will lead to an α% change in Y
  - β: elasticity of Y with respect to K
    - A 1% change in K will lead to a β% change in Y

Math FYI: Cobb Douglas Functions and Logs

Example

$$Y = 2L^{0.75}K^{0.25}$$

- Taking logs:

$$\ln Y = \ln 2 + 0.75 \ln L + 0.25 \ln K$$

- A 1% change in L will yield a 0.75% change in output Y
- A 1% change in K will yield a 0.25% change in output Y

Logarithms in R I

- The `log()` function can easily take the logarithm

```
gapminder <- gapminder %>%
  mutate(loggdp = log(gdpPercap)) # log GDP per capita

gapminder %>% head() # look at it
```

country <fct>	continent <fct>	year <int>
Afghanistan	Asia	1952
Afghanistan	Asia	1957
Afghanistan	Asia	1962
Afghanistan	Asia	1967
Afghanistan	Asia	1972
Afghanistan	Asia	1977

6 rows | 1-3 of 9 columns

Logarithms in R II

- Note, `log()` by default is the **natural logarithm** $ln()$, i.e. base `e`
  - Can change base with e.g. `log(x, base = 5)`
  - Some common built-in logs: `log10`, `log2`

```r
log10(100)
```

```
[1] 2
```

```r
log2(16)
```

```
[1] 4
```

```r
log(19683, base=3)
```

```
[1] 9
```

Logarithms in R III

- Note when running a regression, you can pre-transform the data into logs (as I did above), or just add `log()` around a variable in the regression

term <chr>	estimate <dbl>	std.error <dbl>
(Intercept)	-9.100889	1.227674
loggdp	8.405085	0.148762

2 rows | 1-3 of 5 columns

Types of Logarithmic Models

- Three types of log regression models, depending on which variables we log

1. **Linear-log model:** $Y_i = \beta_0 + \beta_1 \ln X_i$
2. **Log-linear model:** $\ln Y_i = \beta_0 + \beta_1 X_i$
3. **Log-log model:** $\ln Y_i = \beta_0 + \beta_1 \ln X_i$

Linear-Log Model

Linear-Log Model: Interpretation

- **Linear-log model** has an independent variable (X) that is logged

$$Y = \beta_0 + \beta_1 \ln X_i$$

$$\beta_1 = \frac{\Delta Y}{\left(\frac{\Delta X}{X}\right)}$$

- **Marginal effect of $\mathbf{X} \rightarrow \mathbf{Y}$: a 1% change in X $\rightarrow$ a $\frac{\beta_1}{100}$ unit change in Y**

Linear-Log Model in R

term <chr>	estimate <dbl>	std.error <dbl>	statistic <dbl>
(Intercept)	-9.100889	1.227674	-7.413117
loggdp	8.405085	0.148762	56.500206

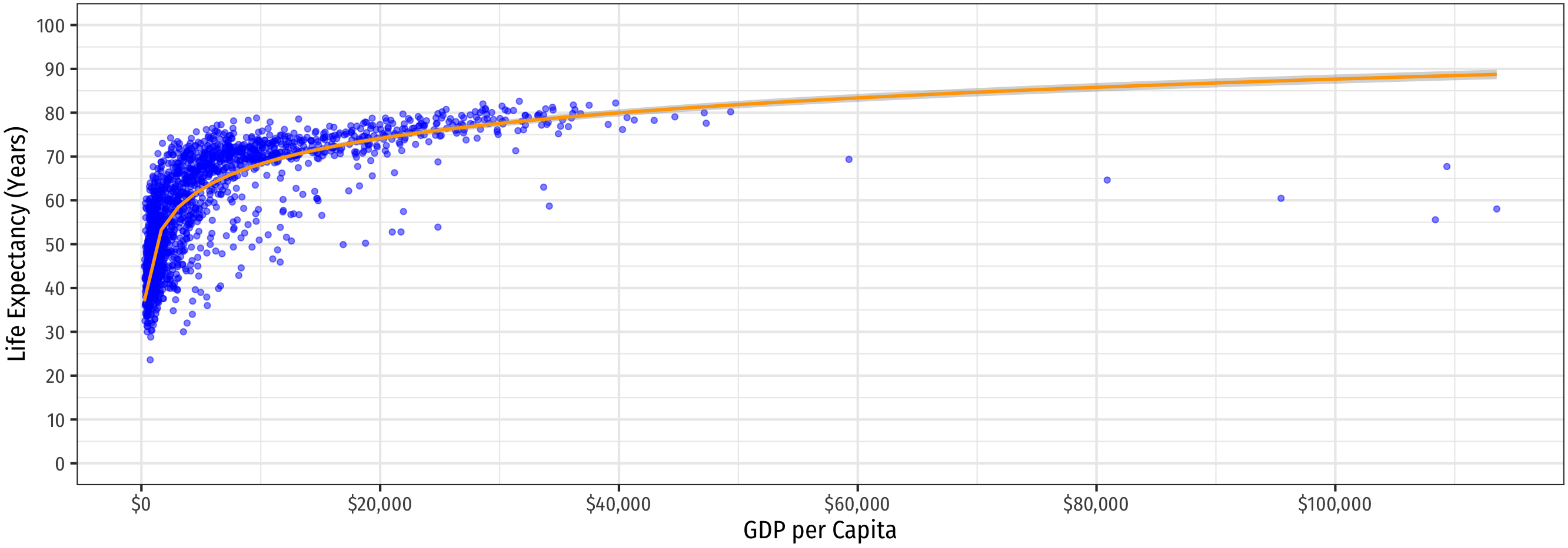
2 rows | 1-4 of 5 columns

$$\widehat{\text{Life Expectancy}}_i = -9.10 + 8.41 \ln \text{GDP}_i$$

- A **1% change in GDP** $\rightarrow$ a $\frac{9.41}{100}$ = **0.0841 year increase** in Life Expectancy
- A **25% fall in GDP** $\rightarrow$ a (-25×0.0841) = **2.1025 year *decrease*** in Life Expectancy
- A **100% rise in GDP** $\rightarrow$ a (100×0.0841) = **8.4100 year increase** in Life Expectancy

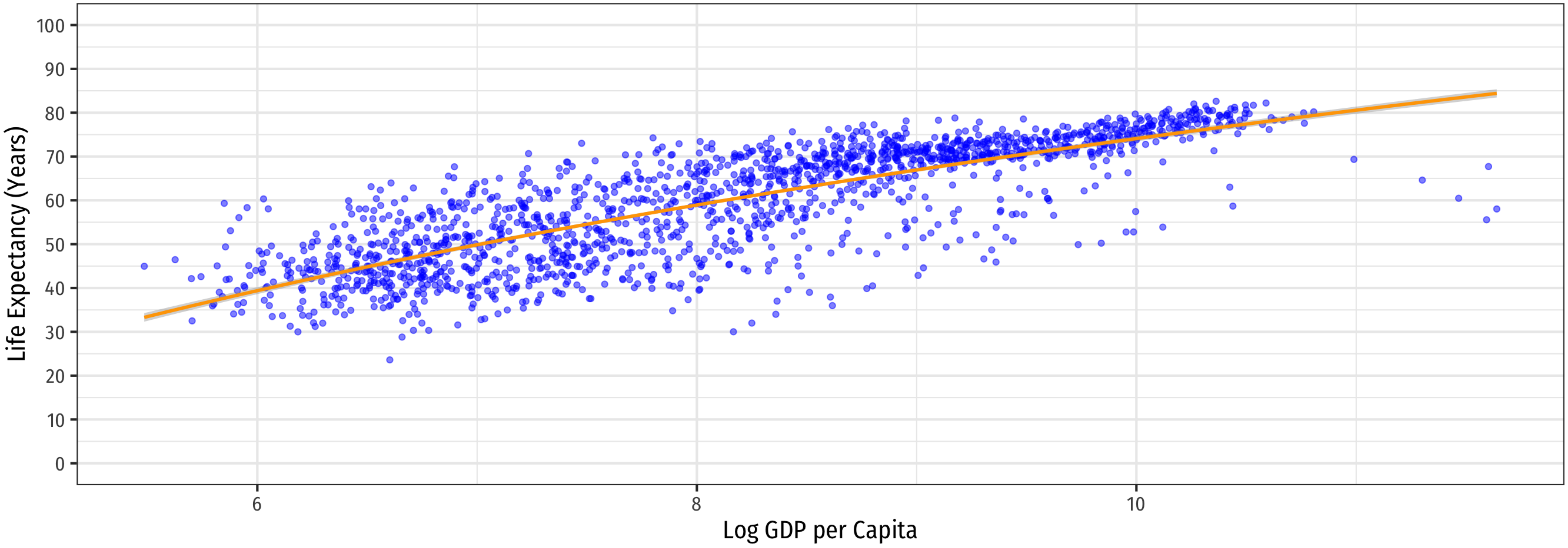
Linear-Log Model Graph (Linear X-Axis)

- Code



Linear-Log Model Graph (Log X-Axis)

▶ Code



Log-Linear Model

Log-Linear Model: Interpretation

- **Log-linear model** has the dependent variable (Y) logged

$$\ln Y_i = \beta_0 + \beta_1 X$$

$$\beta_1 = \frac{\left(\frac{\Delta Y}{Y}\right)}{\Delta X}$$

- **Marginal effect of $\mathbf{X} \rightarrow \mathbf{Y}$: a 1 unit change in X $\rightarrow$ a $\beta_1 \times 100$ % change in Y**

Log-Linear Model in R (Preliminaries)

- We will again have very large/small coefficients if we deal with GDP directly, again let's transform `gdpPercap` into $1,000s, call it `gdp_t`
- Then log LifeExp

```r
gapminder <- gapminder %>%
  mutate(gdp_t = gdpPercap/1000, # first make GDP/capita in $1000s
         loglife = log(lifeExp)) # take the log of LifeExp
gapminder %>% head() # look at it
```

country <fct>	continent <fct>	year <int>
Afghanistan	Asia	1952
Afghanistan	Asia	1957
Afghanistan	Asia	1962
Afghanistan	Asia	1967
Afghanistan	Asia	1972

country <fct>	continent <fct>	year <int>
Afghanistan	Asia	1977

6 rows | 1-3 of 11 columns

Log-Linear Model in R

term	estimate	std.error	statistic
<chr>	<dbl>	<dbl>	<dbl>
(Intercept)	3.966639	0.0058345501	679.85339
gdp_t	0.012917	0.0004777072	27.03958

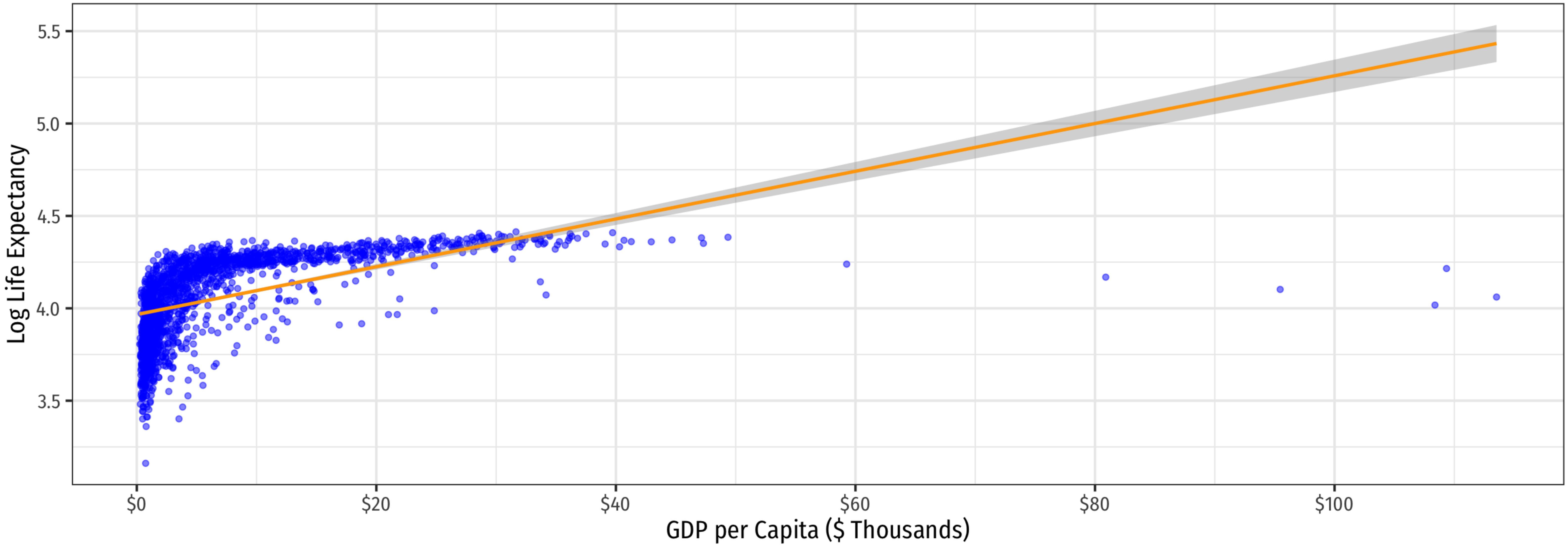
2 rows | 1-4 of 5 columns

$$\ln \widehat{\text{Life Expectancy}}_i = 3.967 + 0.013 \, \text{GDP}_i$$

- A **$1 (thousand) change in GDP** → a $0.013 \times 100\% = $ **1.3% increase** in Life Expectancy
- A **$25 (thousand) fall in GDP** → a $(-25 \times 1.3\%) = $ **32.5% decrease** in Life Expectancy
- A **$100 (thousand) rise in GDP** → a $(100 \times 1.3\%) = $ **130% increase** in Life Expectancy

Linear-Log Model Graph

- Code



Log-Log Model

Log-Log Model

- **Log-log model** has both variables (X and Y) logged

$$\ln Y_i = \beta_0 + \beta_1 \ln X_i$$
$$\beta_1 = \frac{\left(\frac{\Delta Y}{Y}\right)}{\left(\frac{\Delta X}{X}\right)}$$

- **Marginal effect of $\mathbf{X} \rightarrow \mathbf{Y}$: a 1% change in $X \rightarrow$ a β_1 % change in Y**
- β_1 is the **elasticity** of Y with respect to X!

Log-Log Model in R

term <chr>	estimate <dbl>	std.error <dbl>	statistic <dbl>
(Intercept)	2.864177	0.02328274	123.01718
loggdp	0.146549	0.00282126	51.94452

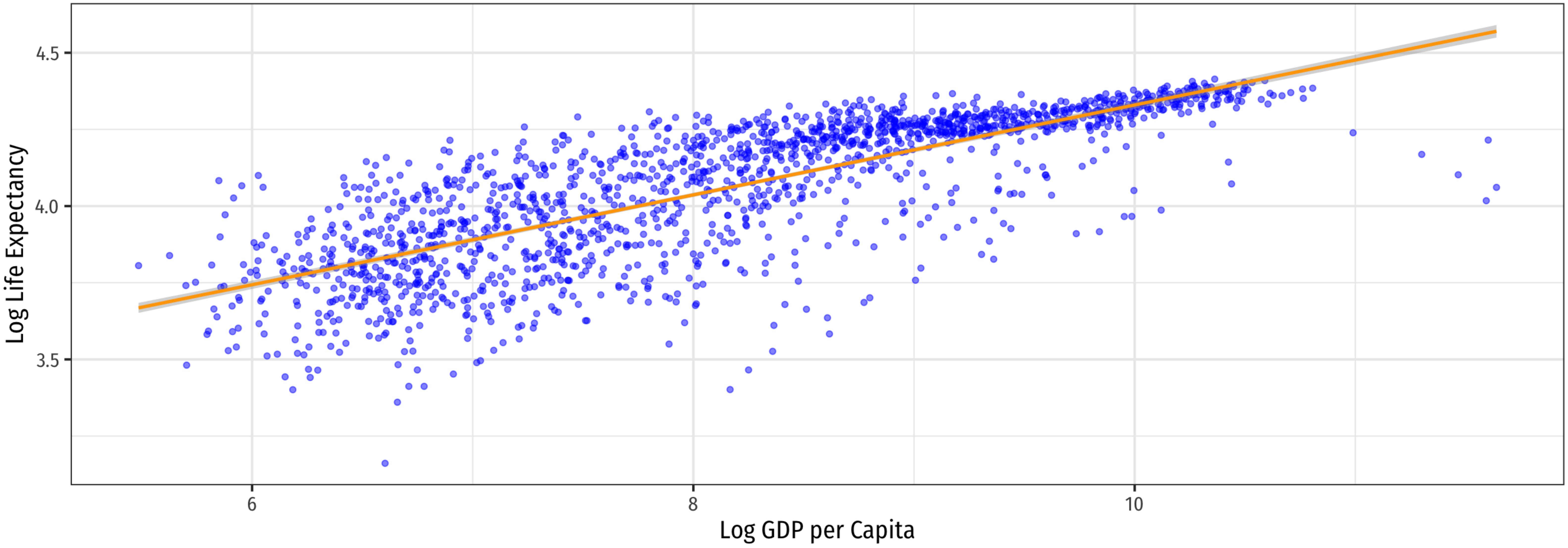
2 rows | 1-4 of 5 columns

$$\widehat{\ln \text{Life Expectancy}_i} = 2.864 + 0.147 \ln \text{GDP}_i$$

- A **1% change in GDP** → a **0.147% increase** in Life Expectancy
- A **25% fall in GDP** → a $(-25 \times 0.147\%)$ = **3.675% decrease** in Life Expectancy
- A **100% rise in GDP** → a $(100 \times 0.147\%)$ = **14.7% increase** in Life Expectancy

Log-Log Model Graph

▶ Code



Comparing Log Models I

Model	Equation	Interpretation
Linear-**Log**	$Y = \beta_0 + \beta_1 \ln X$	1**%** change in $X \rightarrow \frac{\hat{\beta}_1}{100}$ **unit** change in Y
Log-Linear	$\ln Y = \beta_0 + \beta_1 X$	1 **unit** change in $X \rightarrow \hat{\beta}_1 \times 100$**%** change in Y
Log-**Log**	$\ln Y = \beta_0 + \beta_1 \ln X$	1**%** change in $X \rightarrow \hat{\beta}_1$ **%** change in Y

- Hint: the variable that gets **logged** changes in **percent** terms, the **linear** variable (not logged) changes in **unit** terms
  - Going from units $\rightarrow$ percent: multiply by 100
  - Going from percent $\rightarrow$ units: divide by 100

Comparing Models II

▶ Code

	Life Exp.	Log Life Exp.	Log Life Exp.
Constant	−9.10***	3.97***	2.86***
	(1.23)	(0.01)	(0.02)
Log GDP per Capita	8.41***		0.15***
	(0.15)		(0.00)
GDP per capita ($1,000s)		0.01***	
		(0.00)	
n	1704	1704	1704
Adj. R^2	0.65	0.30	0.61
SER	7.62	0.19	0.14

* p < 0.1, ** p < 0.05, *** p < 0.01

- Models are very different units, how to choose?
  1. Compare intuition
  2. Compare R^2's
  3. Compare graphs

Comparing Models III

Linear-Log	Log-Linear	Log-Log
$\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_i$	$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 X_i$	$\ln Y_i = \hat{\beta}_0 + \hat{\beta}_1 \ln X_i$
$R^2 = 0.65$	$R^2 = 0.30$	$R^2 = 0.61$

When to Log?

- In practice, the following types of variables are usually logged:
  - Variables that must always be **positive** (prices, sales, market values)
  - **Very large** numbers (population, GDP)
  - Variables we want to talk about as **percentage changes or growth rates** (money supply, population, GDP)
  - Variables that have **diminishing returns** (output, utility)
  - Variables that have nonlinear scatterplots

- *Avoid* logs for:
  - Variables that are less than one, decimals, 0, or negative
  - Categorical variables (season, gender, political party)
  - Time variables (year, week, day)

Standardizing & Comparing Across Units

Comparing Coefficients of Different Units I

$$\hat{Y}_i = \beta_0 + \beta_1 X_1 + \beta_2 X_2$$

- We often want to compare coefficients to see which variable X_1 or X_2 has a bigger effect on Y
- What if X_1 and X_2 are different units?

Example

$$\widehat{\text{Salary}_i} = \beta_0 + \beta_1 \text{ Batting average}_i + \beta_2 \text{ Home runs}_i$$
$$\widehat{\text{Salary}_i} = -2{,}869{,}439.40 + 12{,}417{,}629.72 \text{ Batting average}_i + 129{,}627.36 \text{ Home runs}_i$$

Comparing Coefficients of Different Units II

- An easy way is to **standardize**[1] the variables (i.e. take the Z-score)

$$X_Z = \frac{X_i - \overline{X}}{sd(X)}$$

- Note doing this will make the constant 0, as both distributions of X and Y are now centered at 0.

1. Also called "centering" or "scaling"

Comparing Coefficients of Different Units: Example

Variable	Mean	Std. Dev.
Salary	$2,024,616	$2,764,512
Batting Average	0.267	0.031
Home Runs	12.11	10.31

$$\widehat{\text{Salary}_i} = -2{,}869{,}439.40 + 12{,}417{,}629.72 \text{ Batting average}_i + 129{,}627.36 \text{ Home runs}_i$$
$$\widehat{\text{Salary}_Z} = 0.00 + 0.14 \text{ Batting average}_Z + 0.48 \text{ Home runs}_Z$$

- **Marginal effects** on Y (in *standard deviations* of Y) from 1 *standard deviation* change in X:
- $\hat{\beta}_1$: a 1 standard deviation increase in Batting Average increases Salary by 0.14 standard deviations

$$0.14 \times \$2,764,512 = \$387,032$$

- $\hat{\beta}_2$: a 1 standard deviation increase in Home Runs increases Salary by 0.48 standard deviations

$$0.48 \times \$2,764,512 = \$1,326,966$$

Standardizing in R

Variable	Mean	SD
`LifeExp`	59.47	12.92
`gdpPercap`	$7215.32	$9857.46

- Use the `scale()` command inside `mutate()` function to standardize a variable

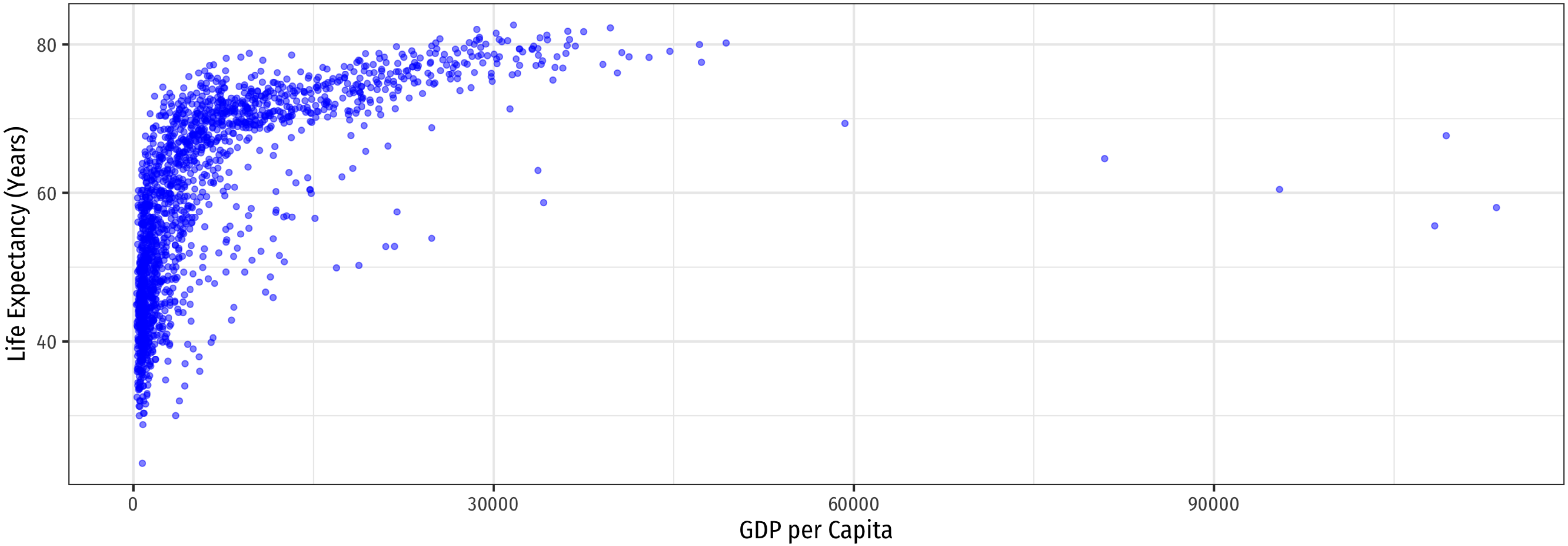
▶ Code

term <chr>	estimate <dbl>
(Intercept)	1.095650e-16
gdp_Z	5.837062e-01

2 rows | 1-2 of 5 columns

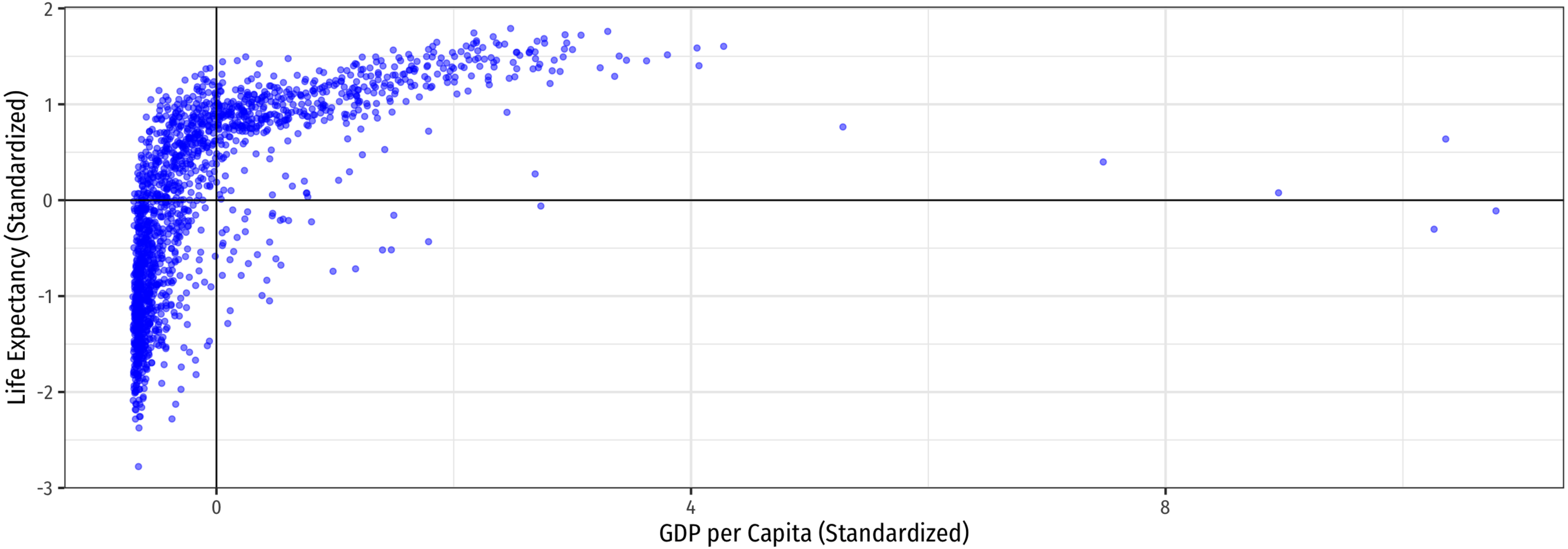
Rescaling: Visually

- ▶ Code



Rescaling: Visually

- Code



Rescaling: Visually

- Both X and Y now have means of 0 and sd of 1

▶ Code

mean_gdp <dbl>	sd_gdp <dbl>	mean_life <dbl>
0	1	0

1 row | 1-3 of 4 columns

Joint Hypothesis Testing

Joint Hypothesis Testing I

Example

Return again to:

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i + \hat{\beta}_2 \text{Northeast}_i + \hat{\beta}_3 \text{ Midwest}_i + \hat{\beta}_4 \text{ South}_i$$

- Maybe region doesn't affect wages *at all*?
- $H_0 : \beta_2 = 0, \; \beta_3 = 0, \; \beta_4 = 0$
- This is a **joint hypothesis** (of multiple parameters) to test

Joint Hypothesis Testing II

- A **joint hypothesis** tests against the null hypothesis of a value for **multiple** parameters:

$$\mathbf{H_0 : \beta_1 = \beta_2 = 0}$$

the hypotheses that **multiple** regressors are equal to zero (have no causal effect on the outcome)

- Our **alternative hypothesis** is that:

$$H_1 : \text{ either } \beta_1 \neq 0 \text{ or } \beta_2 \neq 0 \text{ or both}$$

or simply, that H_0 is not true

Types of Joint Hypothesis Tests

1. $H_0: \beta_1 = \beta_2 = 0$
   - Testing against the claim that multiple variables don't matter
   - Useful under high multicollinearity between variables
   - H_a: at least one parameter $\neq 0$

2. $H_0: \beta_1 = \beta_2$
   - Testing whether two variables matter the same
   - Variables must be the same units
   - $H_a : \beta_1 (\neq, <, \text{ or } >) \beta_2$

3. H_0 : ALL β's $= 0$
   - The "**Overall F-test**"
   - Testing against claim that regression model explains *NO* variation in Y

Joint Hypothesis Tests: F-statistic

- The **F-statistic** is the test-statistic used to test joint hypotheses about regression coefficients with an **F-test**

- This involves comparing two models:

  1. **Unrestricted model**: regression with all coefficients
  2. **Restricted model**: regression under null hypothesis (coefficients equal hypothesized values)

- F is an **analysis of variance (ANOVA)**
  - essentially tests whether R^2 increases statistically significantly as we go from the restricted model→unrestricted model

- F has its own distribution, with *two* sets of degrees of freedom

Joint Hypothesis F-test: Example I

Example

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i + \hat{\beta}_2 \text{Northeast}_i + \hat{\beta}_3 \text{ Midwest}_i + \hat{\beta}_4 \text{ South}_i$$

- $H_0 : \beta_2 = \beta_3 = \beta_4 = 0$
- H_a: H_0 is not true (at least one $\beta_i \neq 0$)

Joint Hypothesis F-test: Example II

Example

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i + \hat{\beta}_2 \text{Northeast}_i + \hat{\beta}_3 \text{ Midwest}_i + \hat{\beta}_4 \text{ South}_i$$

- **Unrestricted model**:

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i + \hat{\beta}_2 \text{Northeast}_i + \hat{\beta}_3 \text{ Midwest}_i + \hat{\beta}_4 \text{ South}_i$$

- **Restricted model**:

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i +$$

Calculating the F-statistic

$$F_{q,(n-k-1)} = \frac{\left(\frac{(R_u^2 - R_r^2)}{q}\right)}{\left(\frac{(1-R_u^2)}{(n-k-1)}\right)}$$

Calculating the F-statistic

$$F_{q,(n-k-1)} = \frac{\left(\frac{(R^2_u - R^2_r)}{q}\right)}{\left(\frac{(1-R^2_u)}{(n-k-1)}\right)}$$

- R^2_u: the R^2 from the **unrestricted model** (all variables)

Calculating the F-statistic

$$F_{q,(n-k-1)} = \frac{\left(\frac{(R^2_u - R^2_r)}{q}\right)}{\left(\frac{(1-R^2_u)}{(n-k-1)}\right)}$$

- R^2_u: the R^2 from the **unrestricted model** (all variables)
- R^2_r: the R^2 from the **restricted model** (null hypothesis)

Calculating the F-statistic

$$F_{q,(n-k-1)} = \frac{\left(\frac{(R_u^2 - R_r^2)}{q}\right)}{\left(\frac{(1 - R_u^2)}{(n-k-1)}\right)}$$

- R_u^2: the R^2 from the **unrestricted model** (all variables)
- R_r^2: the R^2 from the **restricted model** (null hypothesis)
- q: number of restrictions (number of $\beta' s = 0$ under null hypothesis)
- k: number of X variables in .hi[unrestricted model] (all variables)
- F has two sets of degrees of freedom:
  - q for the numerator, $(n-k-1)$ for the denominator

Calculating the F-statistic

$$F_{q,(n-k-1)} = \frac{\left(\frac{(R_u^2 - R_r^2)}{q}\right)}{\left(\frac{(1-R_u^2)}{(n-k-1)}\right)}$$

- **Key takeaway**: The bigger the difference between $(R_u^2 - R_r^2)$, the greater the improvement in fit by adding variables, the larger the F!
- This formula is (believe it or not) actually a simplified version (assuming homoskedasticity)
  - I give you this formula to **build your intuition of what F is measuring**

F-test Example I

- We'll use the `wooldridge` package's `wage1` data again

```r
# load in data from wooldridge package
library(wooldridge)
wages <- wage1

# run regressions
unrestricted_reg <- lm(wage ~ female + northcen + west + south, data = wages)
restricted_reg <- lm(wage ~ female, data = wages)
```

F-test Example II

- **Unrestricted model**:

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i + \hat{\beta}_2 \text{Northeast}_i + \hat{\beta}_3 \text{ Midwest}_i + \hat{\beta}_4 \text{ South}_i$$

- **Restricted model**:

$$\widehat{\text{Wage}}_i = \hat{\beta}_0 + \hat{\beta}_1 \text{ Male}_i +$$

- $H_0 : \beta_2 = \beta_3 = \beta_4 = 0$
- $q = 3$ restrictions (F numerator df)
- $n - k - 1 = 526 - 4 - 1 = 521$ (F denominator df)

F-test Example III

- We can use the `car` package's `linearHypothesis()` command to run an F-test:
  - first argument: name of the (unrestricted) regression
  - second argument: vector of variable names (in quotes) you are testing

```r
# load car package for additional regression tools
library(car)
# F-test
linearHypothesis(unrestricted_reg, c("northcen", "west", "south"))
```

	Res.Df <dbl>	RSS <dbl>	Df <dbl>
1	524	6332.194	NA
2	521	6174.831	3

2 rows | 1-4 of 7 columns

- p-value on F-test < 0.05, so we can reject H_0

All F-test I

```
Call:
lm(formula = wage ~ female + northcen + west + south, data = wages)

Residuals:
    Min      1Q  Median      3Q     Max 
-6.3269 -2.0105 -0.7871  1.1898 17.4146 

Coefficients:
            Estimate Std. Error t value Pr(>|t|)    
(Intercept)   7.5654     0.3466  21.827   <2e-16 ***
female       -2.5652     0.3011  -8.520   <2e-16 ***
northcen     -0.5918     0.4362  -1.357   0.1755    
west          0.4315     0.4838   0.892   0.3729    
south        -1.0262     0.4048  -2.535   0.0115 *  
---
Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1

Residual standard error: 3.443 on 521 degrees of freedom
Multiple R-squared:  0.1376,	Adjusted R-squared:  0.131 
```

- Last line of regression output from `summary()` is an **All F-test**
  - H_0 : all $\beta's = 0$
    - the regression explains no variation in Y
  - Calculates an `F-statistic` that, if high enough, is significant (`p-value` < 0.05) enough to reject H_0

All F-test II

- Alternatively, if you use `broom` instead of `summary()`:
  - `glance()` command makes table of regression summary statistics
  - `tidy()` only shows coefficients

```
glance(unrestricted_reg)
```

r.squared <dbl>	adj.r.squared <dbl>	sigma <dbl>	statistic <dbl>
0.1376433	0.1310225	3.442656	20.78959

1 row | 1-4 of 12 columns

- `statistic` is the All F-test, `p.value` next to it is the p-value from the F test